NAME_____

Massachusetts Institute of Technology
Introduction to Plasma Physics

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NOTES: There are 8 pages to the exam, plus one formula sheet. Make sure that your copy is complete.

Each problem is worth 20%, although the problems differ somewhat in complexity. It is a good idea to look through all problems before deciding in which order to attack them!

Problem 1.

The figure shows field lines in a cylindrically symmetric magnetic mirror plasma confinement device.
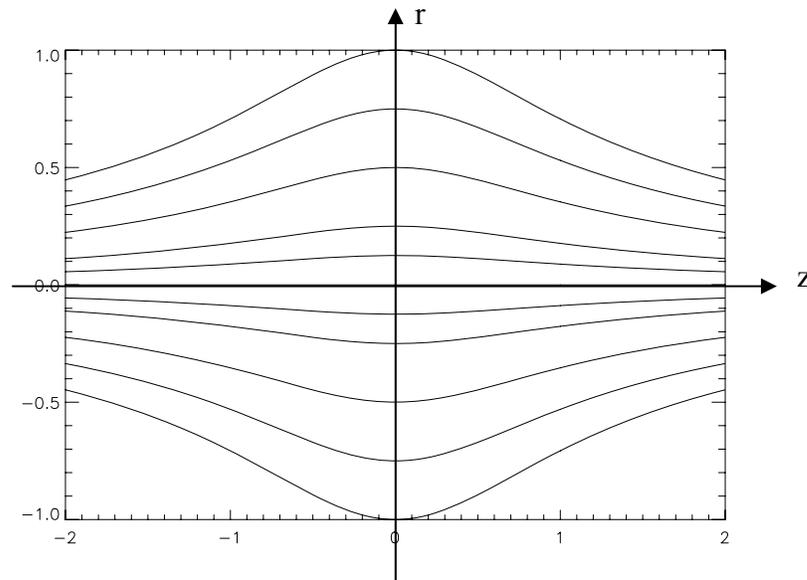
The field is given by

$$\vec{B} = B_0(1+\frac{z^2}{\ell^2})\hat{z} - B_0\frac{rz}{\ell^2}\hat{r}$$

a) A particle orbits in the mirror with its guiding center moving along the z-axis. At z=0, it has total energy $W_0 = W_{\perp 0} + W_{//0}$ where the subscripts $\perp$ and // refer to perpendicular and parallel to the field. Calculate $z(t)$, the motion of the guiding center.

b) Calculate the grad B and curvature components of the drift velocity of the particle when its guiding center is located at

$$z = 0, \, r = \ell / 2$$

c) Find an equation for z_t, the turning point of a particle with a guiding center that passes through the point $z = 0, \, r = \ell / 2$ with energy $W_0 = W_{\perp 0} + W_{//0}$. **Do not solve the equation.** (Hint: this will require calculation of the equation of the field lines.)

Problem 2.

The ideal MHD equations together with the appropriate form of Maxwell's equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \qquad \vec{E} + \vec{v} \times \vec{B} = 0$$

$$\rho \frac{d\vec{v}}{dt} = \vec{j} \times \vec{B} - \nabla p \qquad \nabla \times \vec{B} = \mu_0 \vec{j}, \quad \nabla \cdot \vec{B} = 0 \quad (1)$$

$$\frac{d(p\rho^{-\gamma})}{dt} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

As we have seen in class, the linearized version of these equations supports undamped shear and compressional Alfvén waves in a homogenous plasma immersed in a uniform magnetic field. For example, with $\vec{B} = \hat{z}B_0$ the $\vec{E}$-field in the shear wave can be written

$$\vec{E} = \hat{x}E_1 \cos(k_A z - \omega t)$$

where $k_A = \omega / c_A$ and c_A is the Alfvén speed.

If the conductivity of the plasma is considered to be large but finite, the only modification to the above ideal equations is in the Ohm's law, which takes the form

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$$

As expected, the resistivity leads to damping of the Alfvén wave, as you are asked to show in this problem.

a) Assume the equilibrium plasma is homogenous, and without current or flow. Specifically assume

$$\rho = \rho_0 + \rho_1(\vec{r},t) \qquad \vec{v} = \vec{v}_1(\vec{r},t)$$
$$\vec{B} = \vec{B}_0 + \vec{B}_1(\vec{r},t) \qquad \vec{E} = \vec{E}_1(\vec{r},t)$$
$$\vec{j} = \vec{j}_1(\vec{r},t) \qquad p = p_0 + p_1(\vec{r},t)$$

where the quantities labeled with a subscript 0 are constants while the quantities labeled with subscript 1 are small perturbations. Write down a complete linearized set of equations for the perturbed quantities.

b) Assume that we are interested in waves with space-time dependence given by $\exp(i\vec{k} \cdot \vec{r} - i\omega t)$. In the absence of resistivity, the perturbed magnetic field is given by

$$\vec{B}_1 = -\frac{\vec{k} \times (\vec{v}_1 \times \vec{B}_0)}{\omega}$$

In the presence of resistivity, the perturbed magnetic field can be written in the form

$$\vec{B}_1 = -\frac{\vec{k} \times (\vec{v}_1 \times \vec{B}_0)}{\omega^*}$$

Determine ω^*.

c) Again in the absence of resistivity, the velocity is determined by solution to the equation

$$\omega^2 \vec{V} = c_A^2 \left[\vec{k} \times (\vec{k} \times (\vec{V} \times \hat{z}))\right] \times \hat{z} + c_S^2 \vec{k}\vec{k} \cdot \vec{V} ,$$

where we have taken the equilibrium magnetic field to be in the z-direction. In the presence of resisitivity, the same equation applies, but with $c_A \Rightarrow c_A^*$ and $c_S \Rightarrow c_S^*$. Determine c_A^* and c_S^*.

d) By making the substitution suggested in part c), determine the dispersion relation of the shear Alfvén mode propagating in the direction of $\vec{B}_0$ with $\vec{V} \perp \vec{B}_0$, including the effect of resistivity.

e) Assume that ω is real (corresponding to a source of this frequency.) Then, for the wave considered in part d), $\vec{k} = \hat{z}(k_r + ik_i)$, where k_r and k_i are the real and imaginary parts of the wavenumber. Determine k_r and k_i.

Problem 3.

Consider an axisymmetric θ-pinch in which the plasma rigidly rotates, i.e., $\vec{V} = \hat{\theta}r\Omega$ where Ω is a constant. The central beta $p_0/(B_0^2/2\mu_0)$ is 50%, and the edge speed reaches sonic levels, i.e.,

$a\Omega = \sqrt{\dfrac{2KT_0}{m_i}}$, where T_0 is the central temperature and a is the plasma radius. Throughout this problem

assume $T_e = T_i$ and neglect viscosity.

Note: The two parts of this problem are independent.

a) Find $B(a-)$, the B-field just inside the edge of the plasma in terms of B_0, the field at the center of the plasma. Assume that the plasma pressure vanishes at $r = a$.

b) The plasma temperature as a function of radius is given by $T = T_0(1 - r^2/a^2)^2$ while the density is given by $n = n_0$ where n_0 is a constant. Determine the confinement time of the plasma defined by

$$\tau_E = \frac{W}{P}$$

where W is the thermal energy stored per unit length and P is the power lost by conduction per unit length. Neglect any spatial variation of $\ell n\Lambda$.

Express your answer in terms of: a, the plasma radius; τ_0, a collision time evaluated at $r = 0$; ρ_L, the Larmor radius of a thermal particle also evaluated at $r = 0$; and $B(a-)/B_0$. Be sure to properly identify the collision time and the Larmor radius in terms of type and species.

A page of possibly useful formulas reproduced from the Plasma Formulary is found on the next page.

Problem 4.

Consider 1-D plasma oscillations proportional to $\exp(-i\omega t + ikx)$ in a hot plasma with a 1-D electron distribution function given by

$$\tilde{f}_e(v_x) = \frac{v_e}{\pi} \frac{1}{v_x^2 + v_e^2}.$$

For simplicity assume that k is real, but that ω could be complex.

a) Determine an algebraic dispersion relation for electron oscillations, assuming that the ions are immobile.

b) Solve the dispersion relation obtained in a) for $\omega(k)$.

c) Now assume that the ions have a distribution function given by

$$\tilde{f}_i(v_x) = \frac{v_i}{\pi} \frac{1}{v_x^2 + v_i^2},$$

while the electron distribution function is the same as in part a). Assuming $\omega/k \ll v_e$, determine $\omega(k)$ for ion acoustic waves.

Possibly useful integrals:

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2 + v_e^2)^2} \frac{1}{v_x - \varsigma} dv_x = -\frac{\pi}{2v_e} \left\{ \frac{(\varsigma - iv_e)^2}{(\varsigma^2 + v_e^2)^2} \right\} \quad \text{Im}\varsigma > 0$$

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2 + v_e^2)^2} \frac{1}{v_x - \varsigma} dv_x = -\frac{\pi}{2v_e} \left\{ \frac{(\varsigma + iv_e)^2}{(\varsigma^2 + v_e^2)^2} \right\} \quad \text{Im}\varsigma < 0$$

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2 + v_e^2)} \frac{1}{v_x - \varsigma} dv_x = \pi \left\{ \frac{(v_e + i\varsigma)}{(v_e^2 + \varsigma^2)} \right\} \quad \text{Im}\varsigma > 0$$

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2 + v_e^2)} \frac{1}{v_x - \varsigma} dv_x = \pi \left\{ \frac{(v_e - i\varsigma)}{(v_e^2 + \varsigma^2)} \right\} \quad \text{Im}\varsigma < 0$$

$$\int_{-\infty}^{\infty} \frac{v_x^2}{(v_x^2 + v_e^2)^2} \frac{1}{v_x - \varsigma} dv_x = -\frac{\pi}{2v_e} \varsigma \left\{ \frac{(\varsigma - iv_e)^2}{(\varsigma^2 + v_e^2)^2} \right\} \quad \text{Im}\varsigma > 0$$

$$\int_{-\infty}^{\infty} \frac{v_x^2}{(v_x^2 + v_e^2)^2} \frac{1}{v_x - \varsigma} dv_x = -\frac{\pi}{2v_e} \varsigma \left\{ \frac{(\varsigma + iv_e)^2}{(\varsigma^2 + v_e^2)^2} \right\} \quad \text{Im}\varsigma < 0$$

Problem 5.

A plane wave is incident on a plasma slab, as indicated schematically in the figure below.
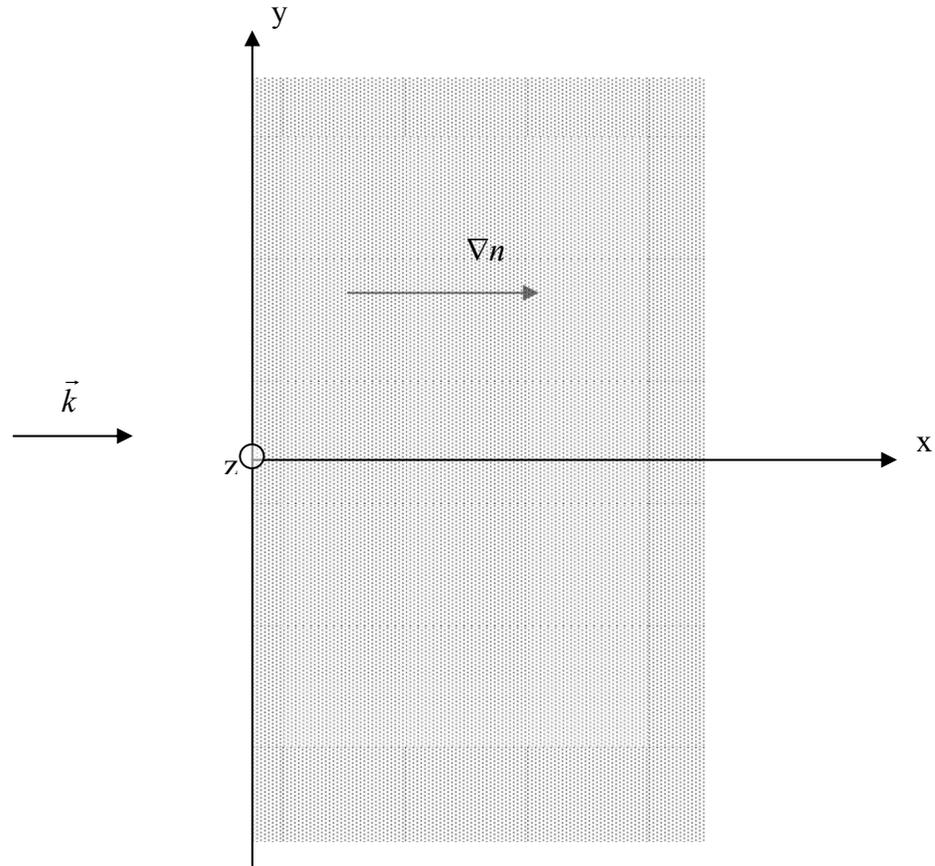
The plasma density is uniform an the y- and z-directions, but varies in the x-direction and is given by

$$n = n_0 \frac{x}{\ell}$$

where ℓ, the gradient scale length, is long compared to the wavelength. i.e., $k\ell \gg 1$. There is a magnetic field in the z-direction, $\vec{B} = \hat{z}B_0$ where B_0 is constant. The parameters are such that

$$\omega_{pe}(n = n_0) = \omega_{ce}.$$

Throughout this problem, consider only the electron response to the wave field.

a) On the left graph below, indicate the region(s) of propagation up to $x/\ell = 4$ for the ordinary mode.

b) On the right graph below, indicate the region(s) of propagation up to $x/\ell = 4$ for the extraordinary mode.

c) What is the maximum depth of propagation of the extraordinary mode for frequencies less than the electron cyclotron frequency?