22.616 Plasma Transport Theory

Prob Set # 1 Solutions

1. Fusion Transport esimates:

$$D = \nu_{ei} \rho_{ep}^2 f_{tr} = \nu_{ei} \frac{V_{te}^2}{\Omega_{ep}^2} f_{tr}$$

$$= \nu_{ei} \frac{V_{te}^2}{\Omega_e^2} \frac{q^2}{\varepsilon^2} \sqrt{\varepsilon} = \underline{\frac{1}{\tau_{ei}} \rho_e^2 \frac{q^2}{\varepsilon^2} \sqrt{\varepsilon}}$$

$$\chi = \nu_{ii} \rho_{ip}^2 f_{tr} = \sqrt{\frac{m_e}{m_i}} \nu_{ei} \frac{m_i}{m_e} \frac{V_{te}^2}{\Omega_e^2} \frac{q^2}{\varepsilon^2} \sqrt{\varepsilon}$$

$$= \sqrt{\frac{m_i}{m_e}} \frac{1}{\tau_{ei}} \rho_e^2 \frac{q^2}{\varepsilon^2} \sqrt{\varepsilon} = \underline{\sqrt{\frac{m_i}{m_e}} D}$$

where

$$\tau_{ei} = 3.44 \times 10^{11} \frac{1}{n(m^{-3})} T(eV)^{3/2} \frac{1}{Z_i \ln\Lambda} \qquad \text{(text book)}$$

$$= 3.44 \times 10^{11} \frac{1}{10^{20}} \times (2\times10^4)^{3/2} \times \frac{1}{16} = \underline{6.08 \times 10^{-4} \text{ (sec)}}$$

$$\rho_e = 2.38\, T_e^{1/2} B^{-1} \Rightarrow \qquad \text{(NRL Formulary)}$$

$$= 2.38 \times (10^4 \times 2)^{\frac{1}{2}} \frac{1}{5\times10^4}$$

$$= 6.73 \times 10^{-3} \text{ (cm)} \qquad \text{(For toroidal field)}$$

Choose $\varepsilon = \frac{a}{R} = 0.33$ then $\frac{q}{\varepsilon} = 5$

$$D = \frac{1}{6.08\times10^{-4}} \times (\cancel{2.38}\ 6.73 \times 10^{-3})^2 \times 5^2 \times \sqrt{0.33}$$

$$= \underline{1.07\ (cm^2/sec)}$$

Then

$$\tau_p = \frac{a^2}{D} = \frac{300^2}{1.07} = \underline{8.41\times10^4}\ (sec)$$

$$\tau_E = \frac{a^2}{\chi} = \sqrt{\frac{m_e}{m_i}}\frac{a^2}{D} = \frac{1}{43}\times 8.41\times10^4 = \underline{1.96\times10^3}\ (sec)$$

The above calculation is based on the assumption of ~~Banana orbi~~ random walk of Banana trapped orbits. Electron diffusion colliding with ions dominates particle transport. Ion ~~heat~~ diffusion dominate heat transport.

Neoclassical Pinch time scale

$$\tau_{pinch} = \frac{a}{V_p} = \frac{aB_p}{E_T} = \frac{2\pi R a B_p}{V_T}$$

$$= \frac{2\pi \times 9 \times 3 \times 1}{0.02} = 8.5 \times 10^3 \text{ sec}.$$

Therefore, the heat diffusion effect is the strongest of all.

2. Diffusion equation solution and properties

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Then $\frac{\partial P}{\partial t} = -\frac{1}{2t} P + \frac{x^2}{4Dt^2} P$

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{x}{2Dt} P\right) = -\frac{1}{2Dt} P + \frac{x^2}{4D^2t^2} P$$

So $\boxed{\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}}$

$$P(x,0) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

Also $\int_{-\infty}^{\infty} P(x,0)\,dx = \lim_{t\to 0}\int_{-\infty}^{\infty} P(x,t)\,dx = \lim_{t\to 0} 1 = 1$

Therefore $P(x,t)$ also satisfy the initial condition:
$P(x,0) = \delta(x)$

a). $\int_{-\infty}^{\infty} dx\, P(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} dx$ (let $y = \frac{x}{\sqrt{4Dt}}$)

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$ (let $z = \sqrt{y}$

$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} \frac{1}{2} z^{-\frac{1}{2}} dz$

$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1$$

b). Because $x P(x,t)$ is an odd function about x. then

$$\int_{-\infty}^{\infty} P(x,t)\, x\, dx = \langle x \rangle = 0$$

c) $$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 P(x,t)$$

$$= \int_{-\infty}^{\infty} dx\, x^2 \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}} \qquad \left(\text{let } y = \frac{x}{\sqrt{4Dt}}\right)$$

$$= \frac{8Dt}{\sqrt{\pi}} \int_0^{\infty} dy\, y^2 e^{-y^2} \qquad (z = \sqrt{y})$$

$$= \frac{4Dt}{\sqrt{\pi}} \int_0^{\infty} dz\, z^{\frac{1}{2}} e^{-z}$$

$$= \frac{4Dt}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \underline{2Dt}.$$

we use the property of Gamma Function

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots \frac{1}{2}\sqrt{\pi}$$

3. Diffusion ~~of~~ Equation & Green's Function

As stated

$$G(x,x',t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-x')^2}{4Dt}}$$

Satisfy

$$\begin{cases} \frac{\partial G}{\partial t} = D\frac{\partial^2 G}{\partial x^2}, & -\infty < x < \infty, t > 0 \\ G(x,x',0) = \delta(x-x') \end{cases}$$

Let $T(x,t) = \int_{-\infty}^{\infty} dx' G(x,x',t) T_0(x')$, then

$$\frac{\partial T}{\partial t} = \int dx' \frac{\partial G}{\partial t} T(x')$$

$$= \int dx' D\frac{\partial^2 G}{\partial x^2} T(x')$$

$$= D\frac{\partial T}{\partial x^2}$$

So $T(x,t)$ satisfy the diffusion equation.

Also $T(x,0) = \int_{-\infty}^{\infty} dx' G(x,x',0) T_0(x')$

$$= \int_{-\infty}^{\infty} dx' \cancel{G(x} \delta(x-x') T_0(x')$$

$$= T_0(x)$$

Which tells that $T(x,t)$ satisfy initial condition.

When $|x| \to \infty$, $G(x,x',t) \to 0$. So we have

$$T(x,t) \to 0 \quad \text{as} \quad |x| \to \infty \; \& \; t \neq 0$$

4. Solution For Metallic Heat Conduction

we have proved (In prob 3)

$$T(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\chi_0 t}}e^{-\frac{(x-x')^2}{4\chi_0 t}}T(x',0)\,dx'$$

satisfy
$$\begin{cases} T_t = \chi_0 T_{xx}, & -\infty<x<\infty,\ t>0 \\ T(x,0)=T_0(x) \end{cases}$$

Now ~~wan~~ we want to solve a half-infinite domain problem

$$\frac{\partial T}{\partial t}=\chi_0\frac{\partial^2 T}{\partial x^2},\quad 0<x<\infty,\ 0<t<\infty$$

$$T(x,0)=0,\quad 0<x<+\infty$$

$$T(0,t)=T_H,\quad 0<t<\infty$$

Make up a function for $T(x,0)$ to extend the ~~prob~~ initial condition to the whole space of x $(-\infty<x<\infty)$. i.e.

$$T(x,0)=\begin{cases} 0, & x>0 \\ 2T_H, & x<0 \end{cases}$$

Then $$T(x,t)=\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\chi_0 t}}e^{-\frac{(x-x')^2}{4\chi_0 t}}2T_H\,dx'$$

Hence $$T(0,t)=2\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\chi_0 t}}e^{-\frac{x'^2}{4\chi_0 t}}T_H\,dx'$$

$$=T_H$$

which satisfy the boundary condition of this problem Automatically

So. we have

$$T(x,t) = 2T_H \int_{\infty}^{0} \frac{1}{\sqrt{4\pi\chi_0 t}} e^{-\frac{(x-x')^2}{4\chi_0 t}} dx' \qquad \left(\text{Let } y = \frac{x'-x}{\sqrt{4\chi_0 t}}\right)$$

$$= T_H \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\chi_0 t}}}^{\infty} e^{-y^2} dy$$

$$= \underline{T_H \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\chi_0 t}}\right)\right)}$$

5. Monte Carlo Solution:

$$\langle R_n^2 \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} P(R_n) R_n^2 \, dR_n$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} R_n^2 \, dR_n$$

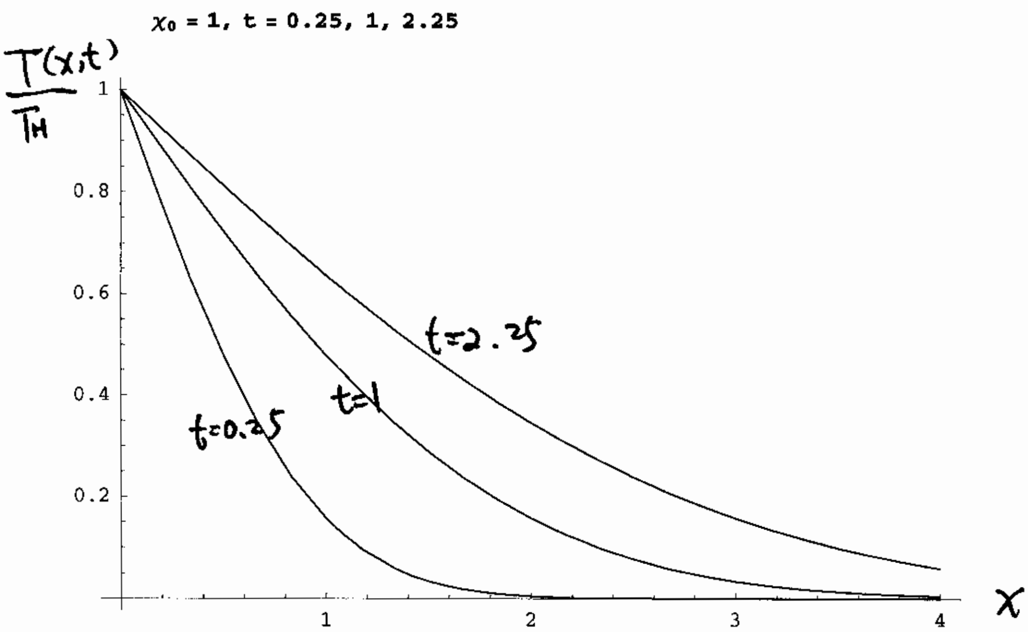
$$= 2 \int_0^{\frac{1}{2}} R_n^2 \, dR_n$$

$$= \frac{1}{12}$$

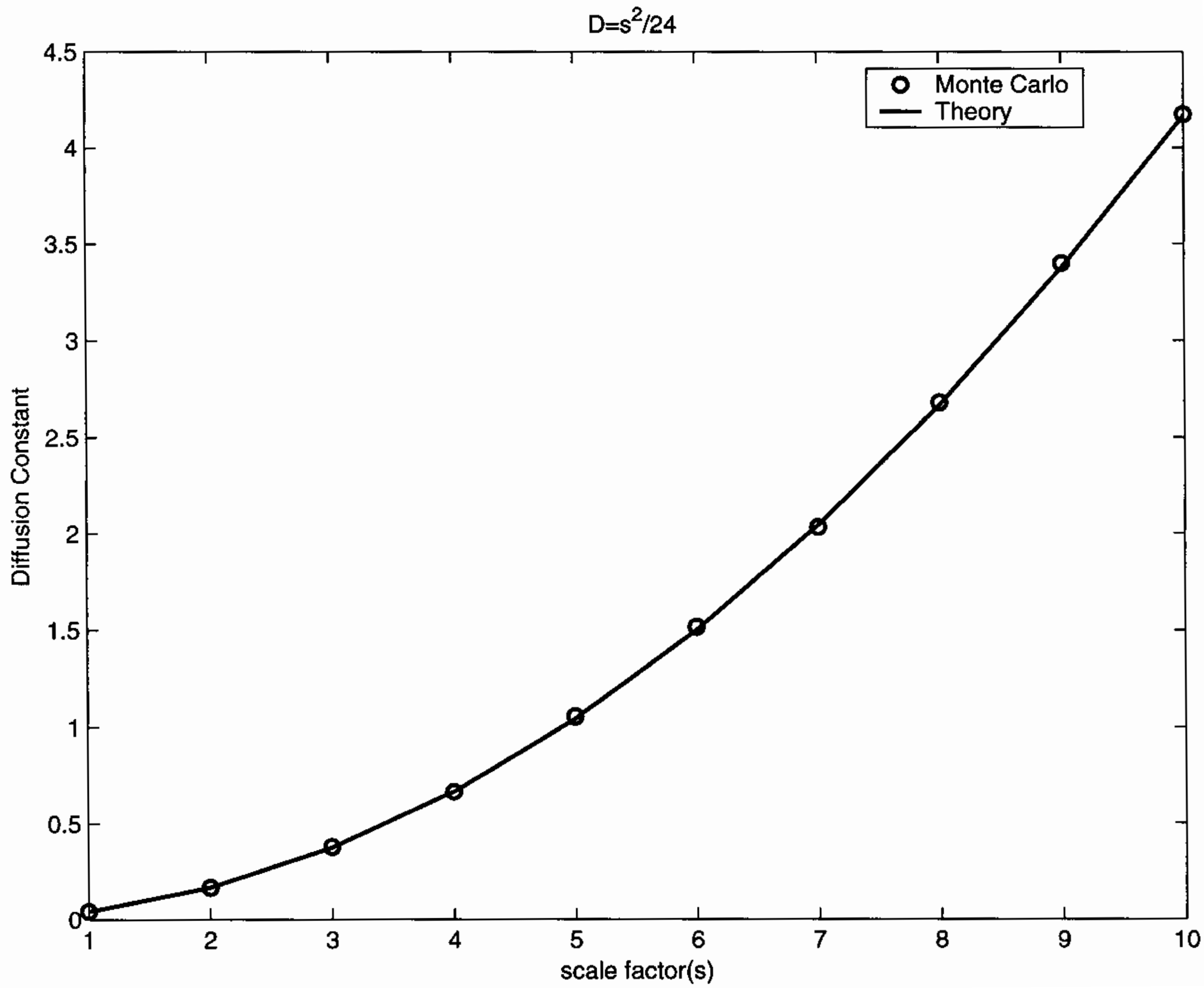
since $s^2 \langle R_n^2 \rangle = 2 \quad \Rightarrow \quad \underline{s = \sqrt{24}}$

Also we have $\langle x^2 \rangle \equiv N \langle \delta x_n^2 \rangle = N s^2 \langle R_n^2 \rangle = \frac{N s^2}{12}$

$$= 2DN \quad \Rightarrow \quad \underline{D = \frac{s^2}{24}}$$



$$T(x,t) = T_H \left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\chi_0 t}}\right)\right)$$



D=s^2/24
Monte Carlo
Theory
Diffusion Constant
scale factor(s)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
1
2
3
4
5
6
7
8
9
10