Plasma Transport theory

problem Set #4   Solutions

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1. Collisional Guiding Center Scattering

$$C_{ei}^R(f_e) = \left\langle \frac{1}{\Omega_e} b \times \nabla_R \cdot \frac{\nu_{ei}}{2} v_{Te}^3 \underline{\underline{U}} \cdot \frac{1}{\Omega_e} b \times \nabla_R f_e \right\rangle_\phi$$

(Assume $f_e = f_e(\underline{R}, \phi, \underline{v})$)

$$= \left\langle \frac{\nu_{ei} v_{Te}^2}{2\Omega_e^3} \frac{1}{2\pi} \oint d\phi \right\rangle \nabla_R$$

$$= \nabla_R \, D(v_\parallel, v_\perp) \, \nabla_R f_e$$

with $$D(v_\parallel, v_\perp) = \left\langle \nu_{ei} \frac{v_{Te}^3}{2\Omega_e^2} e_y \cdot \underline{\underline{U}} \cdot e_y \right\rangle_\phi$$

(An simplified version is considered here. Assume y-direction homogeneous)

$$= \frac{1}{2\pi} \frac{\nu_{ei} v_{Te}^3}{2\Omega_e^2} \oint d\phi \frac{e_y}{v} \left(\underline{\underline{I}} - \frac{v v}{v^2} \right) \cdot e_y$$

$$= \frac{\nu_{ei} v_{Te}^3}{2\Omega_e^2} \frac{1}{2\pi} \oint d\phi \frac{1}{v} \left(1 - \frac{v_y v_y}{v^2} \right)$$

$$= \frac{\nu_{ei} v_{Te}^3}{2\Omega_e^2} \frac{1}{2\pi} \oint d\phi \frac{1}{v^3} \left(v^2 - v_\perp^2 \sin^2\phi \right)$$

$$= \frac{\nu_{ei} v_{Te}^3}{2\Omega_e} \frac{1}{v^3} \left(v^2 - \frac{1}{2} v_\perp^2 \right)$$

$$\underline{= \cdot \frac{\nu_{ei} v_{Te}^3}{2\Omega_e} \frac{1}{v^3} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)}$$

$$\begin{cases} v^2 = v_\perp^2 + v_\parallel^2 \\ v_x = v_\perp \cos\phi \\ v_y = v_\perp \sin\phi \end{cases}$$

Let $\mu = \cos\theta$, then

$$C_{ei}^{v}(f_e) = \nu_{ei}\frac{v_{Te}^3}{v^3}\frac{1}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}f_e$$

2. Diamagnetic Flow

$$f_M(\underline{R}) = f_M(\underline{r}-\underline{\rho}) = f_M(\underline{r}) - \underline{\rho}\cdot\nabla f(\underline{r})$$

First order approximation.

Then

$$n\underline{u}^* = \int d^3v\,\underline{v}\,f_M(\underline{R})$$

$$= \int d^3v\,\underline{v}\,(f_M(\underline{r}) - \underline{\rho}\cdot\nabla f_M(\underline{r}))$$

$$= -\int d^3v\,\underline{\rho}\cdot\nabla f_M(\underline{r})$$

$$= -\int d^3v\,\frac{\underline{b}\times\underline{v}}{\Omega_e}\cdot(\nabla\ln p_e + (\frac{mv^2}{2T_e}-\frac{5}{2})\nabla\ln T_e)f_M\,\underline{v}$$

$$\equiv \frac{1}{\Omega_e}\underline{b}\times(\nabla\ln p_e \mp f$$

$$= \frac{1}{\Omega_e}\underline{b}\times\nabla\ln p_e\cdot\int d^3v\,\underline{v}\underline{v}\,f_M + \frac{1}{\Omega_e}\underline{b}\times\nabla\ln T_e\cdot\int d^3v\,\underline{v}\underline{v}(\frac{mv^2}{2T_e}-\frac{5}{2})f_M$$

where

$$\left(\int d^3v\,\underline{v}\underline{v}\,f_M\right)_{ij} = \int d^3v\,v_i v_j\,f_M$$

$$= \delta_{ij} \int d^3v \, v_i^2 \, f_M$$

$$= \delta_{ij} \frac{T_e n_e}{m_e}$$

So $\quad \int d^3v \, \underline{v}\,\underline{v} \, f_M = \frac{n_e T_e}{m_e} \, \underline{\underline{I}}$

Similarly,

$$\int d^3v \, \underline{v}\,\underline{v} \left(\frac{m_e v^2}{2 T_e} - \frac{5}{2} \right) f_M$$

$$= \underline{\underline{I}} \int d^3v \, \frac{v^2}{3} \left(\frac{v^2}{v_{Te}^2} - \frac{5}{2} \right) \frac{n_e}{(\pi v_{Te}^2)^{3/2}} \, e^{-\frac{v^2}{v_{Te}^2}} \qquad ; \qquad v_{Te} \equiv \sqrt{\frac{2 T_e}{m_e}}$$

$$= n_e \underline{\underline{I}} \, \frac{4\pi}{3} \int_0^\infty dv \, v^4 \left(\frac{v^2}{v_{Te}^2} - \frac{5}{2} \right) \frac{1}{v_{Te}^3} \, e^{-\frac{v^2}{v_{Te}^2}}$$

$$= n_e \underline{\underline{I}} \, \frac{4\pi}{3} v_{Te}^2 \int_0^\infty dx \, x^4 \left(x^2 - \frac{5}{2} \right) e^{-x^2} \qquad \left(x \equiv \frac{v}{v_{Te}} \right)$$

$$= n_e \underline{\underline{I}} \, \frac{T_e}{m_e} \, \frac{4\pi}{3} \int_0^\infty dy \, y^{3/2} \left(y - \frac{5}{2} \right) e^{-x^2} \qquad \left(y \equiv x^2 \right)$$

$$= \frac{n_e T_e}{m_e} \, \frac{4\pi}{3} \, \underline{\underline{I}} \left(\Gamma\!\left(\tfrac{5}{2} \right) - \frac{5}{2} \Gamma\!\left(\tfrac{3}{2} \right) \right)$$

$$= 0$$

Therefore we get

$$n u^* = \frac{1}{\Omega_e} \underline{b} \times \ln P_e \cdot \frac{n_e T_e}{m_e} \underline{\underline{I}}$$

$$= \frac{1}{m_e \Omega_e} \underline{b} \times \nabla P_e$$

3. Electron-Ion Temperature Equilibrium

The electron-ion Energy exchange operator is in the form

$$C_{ei}^E (f_e) = \frac{m_e}{2 m_i} v_{Te}^3 \frac{1}{v^2} \frac{\partial}{\partial v} \left(\nu_{ei} (v) \times f_e + \frac{T_i}{m_e} v \frac{\partial}{\partial v} f_e \right)$$

Compare the expression in Eq 3.40 text book. we easily get

$$\nu_{ei}(v) = \hat{\nu}_{ei} \left(\frac{v}{v_{Ti}} \right)^2 4 G \left(\frac{v}{v_{Ti}} \right)$$

with

$$\hat{\nu}_{ei} = \frac{4 \pi n_i Z_i^2 e^4 \ln \Lambda}{m_e^2 v_{Te}^3}$$

If we make the ordering $\frac{m_e}{m_i} \sim \frac{P_e}{L_\perp} \sim \frac{E}{E_R}$, the zeroth order kinetic

equation gives $f_e^0 = f_{max} = \frac{n_e}{(\pi v_{Te}^2)^{3/2}} e^{-\left(\frac{v}{v_{\partial Te}} \right)^2}$

The first order kinetic equation gives

$$v_{||} \underline{b} \cdot \nabla f_e^0 - \frac{e}{m_e} E_{||} \frac{\partial}{\partial v_{||}} f_e^0 = C_e^v (f_e^1) + C_{ei}^E (f_e^0)$$

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Use the properties of C_0^V : conserve particles and Energy

Take the particle and Energy momets of above equation. Notice

the left handside vanishes. the we req have

$$\int d^3v \ C_{ei}^E (f_e^0) = 0 \qquad \text{---} \ 1°$$

$$\int d^3v \ \tfrac{1}{2}mv^2 \ C_{ei}^E (f_e^0) = 0 \qquad \text{---} \ 2°$$

Eq $1°$ Can be verified as :

$$\int d^3v \ C_{ei}^E (f_e^0) = \frac{m_e}{2m_i} \ v_{Te}^3 \ 4\pi \int_0^\infty dv \ v^2 \tfrac{1}{v^2} \tfrac{\partial}{\partial v} \left(\nu_{ei}(v) (f_e^0 + \tfrac{T_i}{m_e} \tfrac{1}{v} \tfrac{\partial f_e^0}{\partial v}) \right]$$

$$= 2\pi \frac{m_e}{m_i} \ v_{Te}^3 \left[\nu_{ei}(f_e^0 + \tfrac{T_i}{m_e} \tfrac{-m v}{T_e} \ v f_e^0) \right]_0^\infty$$

$\nu_{ei}(v) f_e^0 \to 0$, as $v \to \infty$

$\nu_{ei}(v) \to 0$, as $v \to 0$.

$S_o \quad \int d^3v \ C_{ei}^E (f_e^0) = 0$

Eq $2°$ can be evaluated as

$$\int_0^\infty d^3v \, \frac{1}{2} m_e v^2 \, C_{ei}^E (f_e^0)$$

$$= \frac{m_e}{4 m_i} m_e V_{Te}^3 \, 4\pi \int_0^\infty dv \, v^2 v^2 \frac{1}{v^2} \frac{\partial}{\partial v} \left[2\nu_{ei}(v) \left(f_e^0 + \frac{T_i}{m_e} \frac{1}{v} \frac{\partial}{\partial v} f_e^0 \right) \right]$$

$$= -\pi \frac{m_e}{m_i} m_e V_{Te}^3 \int_0^\infty dv \, 2v \, \nu_{ei}(v) \, \partial f_e^0 \left(1 - \frac{T_i}{T_e} \right)$$

$$= 2\pi \frac{m_e}{m_i} m_e V_{Te}^3 \frac{T_i - T_e}{T_e} \int_0^\infty dv \, v \, \nu_{ei}(v) f_e^0$$

$$= 4\pi (T_i - T_e) \frac{m_e}{m_i} V_{Te} \, \underbrace{\int_0^\infty dv \, v \, \nu_{ei}(v) f_e^0}_{\sim \frac{\hat{\nu}_{ei}}{V_{Te}} n_e}$$

$$\sim 4\pi P \hat{\nu}_{ei} (T_i - T_e) \frac{m_e}{m_i}$$

Usually $T_i \neq T_e$, $T_i \neq T_e$, so $\int_0^\infty d^3v \, \frac{1}{2} m_e v^2 \, C_{ei}^E (f_e^0) \neq 0$

i.e., The Integrability of First order kinetic Equation <1> requires

no energy exchange term $C_{ei}^E(f_e^0)$. So $C_{ei}^E (f_e^0) \sim \frac{m_e}{m_i}$

is a higher order (>1) process. If we make $\frac{m_e}{m_i} \sim \frac{\rho_e}{LL} \sim \frac{E}{E_R}$

will leads to an ill-posed transport theory

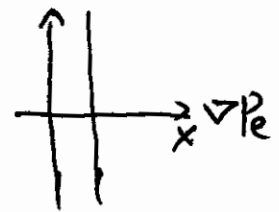The physical basis is that in the first order kinetic equation.

we require the collision operator to conserve Energy for electrons

But the $C_{ei}^E(f_e)$ stands for an energy exchange process between

Ions and electrons. It can not make the energy balance.

4. Flux-Friction Calculation of Radial Flux :

   Momentum balance equation :

   $$0 \simeq -\nabla P_e + m_e \Omega_e n \underline{V} \times \underline{b} + \underline{F}_{ei}$$

1° The x-component gives $\quad 0 \simeq -\underline{e}_x \cdot \nabla P_e + m_e \Omega_e n \, \underline{e}_x \cdot \underline{V} \times \underline{b}$

$$\boxed{n V_x = \frac{\cancel{\partial}}{m_e \Omega_e} \cancel{V_y}}$$

$\Rightarrow \quad V_y = \frac{1}{n m_e \Omega_e} \frac{\partial P_e}{\partial X}$

2° The y-component gives $\quad 0 \simeq m_e \Omega_e n \, \underline{V} \times \underline{b} \cdot \underline{e}_y + \underline{F}_e \cdot \underline{e}_y$

$$n V_x = \frac{1}{m_e \Omega_e} \underline{e}_y \cdot \underline{F}_{ei}$$

$$= \frac{1}{\Omega_e} \underline{e}_y \cdot \int d^3 v \, \underline{v} \, C_{ei}^L(f_e^\circ)$$

Assume f_e^0 to be a drift Maxwellian

$$f_e^0 = \frac{n}{(\pi V_{Te}^2)^{3/2}} e^{-\frac{|\underline{v}-\underline{V}_y|^2}{V_{Te}^2}}$$

Let's first calculate

$$\int d^3v \ \underline{v} \ C_{ei}^L(f_e^0)$$

$$= \int d^3v \ \underline{v} \left(\frac{\partial}{\partial \underline{v}} \cdot \hat{\nu}_{ei} \frac{V_{Te}^3}{2} \underline{U} \cdot \frac{\partial}{\partial \underline{v}} f_e^0 \right)$$

Integrate by parts
$$= -\hat{\nu}_{ei} \frac{V_{Te}^3}{2} \int d^3v \ \underline{U} \cdot \frac{\partial}{\partial \underline{v}} f_e^0$$

$$= + \hat{\nu}_{ei} \frac{V_{Te}^3}{2} \int d^3v \ \frac{1}{v}\left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2} \right) \frac{(\underline{v}-\underline{V}_y) \cdot 2}{V_{Te}^2} f_e^0$$

$$= -\hat{\nu}_{ei} \ V_{Te} \int d^3v \ \frac{1}{v}\left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2} \right) \cdot \underline{V}_y \ f_e^0$$

Notice $V_y = \frac{1}{n m_e \Omega_e} \frac{\partial P_e}{\partial x} \sim \frac{P_e}{n m_e \Omega_e} \frac{1}{L_\perp} \sim V_{Te} \frac{\rho_e}{L_\perp} \ll V_{Te}$

So the zeroth order approximation. which is required for this calculation. we can take

$$f_e^0 \simeq \frac{n}{(\pi V_{Te}^2)^{3/2}} e^{-\frac{v^2}{V_{Te}^2}}$$

Therefore we have

$$n V_x = \frac{-1}{n_e} \underline{e}_y \cdot \int d^3v \; \frac{\hat{\tau}_{ei} v_{te}}{v} \left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2} \right) \cdot \underline{V}_y \, f_e^0$$
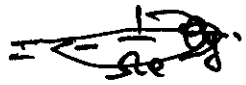
$$= -\frac{\hat{\tau}_{ei}}{n_e} \underline{e}_y \cdot \int d^3v \; \frac{\partial v_{te}}{v} \left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2} \right) f_e^0 \cdot \underline{V}_y$$

$$= -\frac{\hat{\tau}_{ei}}{n_e} \underline{e}_y \cdot \underline{\underline{I}} \; 4\pi \int dv \, v^2 \frac{v_{te}}{v} \left(1 - \frac{1}{3} \frac{v^2}{v^2} \right) f_e^0 \cdot \underline{V}_y$$

$$= -\frac{\hat{\tau}_{ei}}{n_e} \, \underline{e}_y \cdot \underline{e}_y \, \underline{V}_y \, \frac{4\pi}{3} \, \frac{2n}{\pi^{3/2} v_{te}^2} \int_0^\infty dv \, v \, e^{-\left(\frac{v}{v_{te}}\right)^2}$$

$$= -n \frac{\hat{\tau}_{ei}}{n_e} \, \underline{V}_y \, \frac{4\pi}{3\pi^{1/2}}$$

$$= -\frac{n V_y}{n_e \tau_{ei}}$$

$$= -\frac{1}{m_e n_e^2 \tau_{ei}} \frac{\partial p}{\partial x}$$

This is consistent with (Eq 4.17) In textbook.