2. Cross-Product

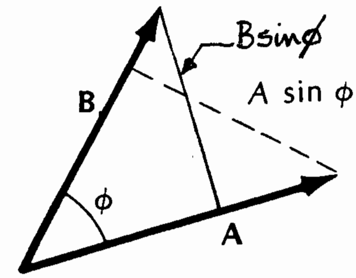
- vector product
- outer product
- result is another vector

$\vec{C} = \vec{A} \times \vec{B}$

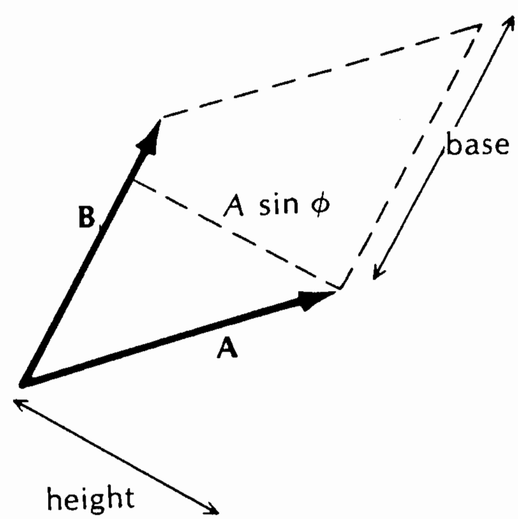
$C = A\underbrace{B \sin\varphi}$ (magnitude)

↑ component of B $\perp A$.

$0 < \varphi < \pi$

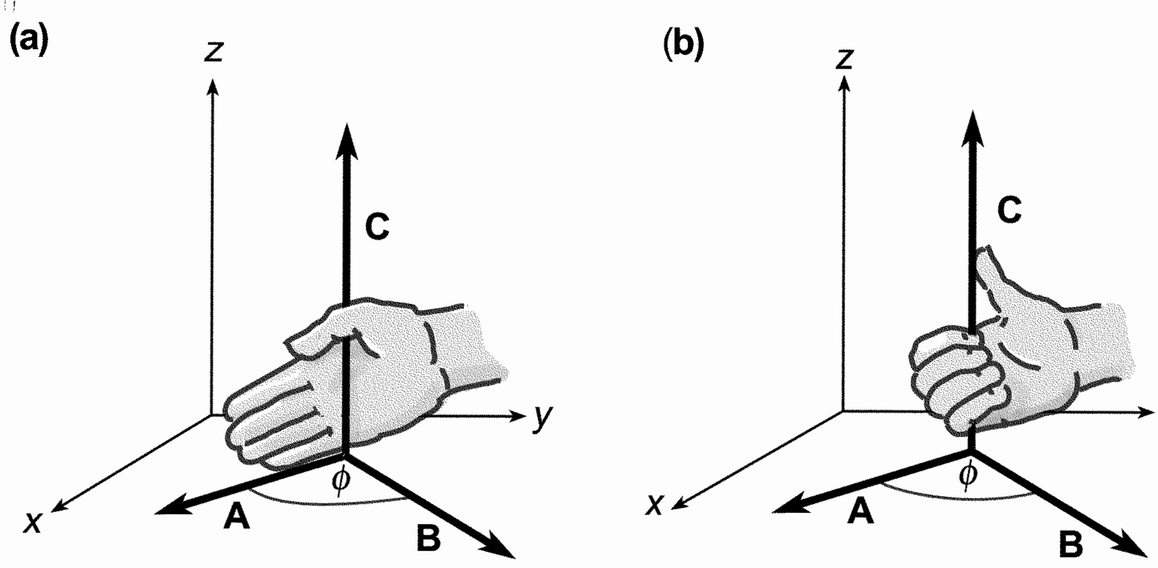


(a)



(b)

(a) $A \sin \phi$ is the component of **A** perpendicular to **B.** (b) $AB \sin \phi$ is the area of the parallelogram.



Right Hand Rule

- Direction of $\vec{C}$ is $\perp$ to plane formed by $\vec{A}$ and $\vec{B}$
- $\therefore \vec{C}$ is $\perp$ to both $\vec{A}$ and $\vec{B}$!!

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$\vec{A}$ and $\vec{B}$ define a plane
$\vec{C}$ is along normal to this plane } RHR

If vectors are parallel:

$$\vec{A} \times \vec{A} = AA \sin 0^\circ$$
$$= 0$$

↑ cross-product is a useful test for vectors which are parallel

$$\left.\begin{array}{l}\hat{i}\times\hat{i}=0\\ \hat{j}\times\hat{j}=0\\ \hat{k}\times\hat{k}=0\end{array}\right\}$$ Parallel unit vectors

$$\left.\begin{array}{l}\hat{i}\times\hat{j}=-\hat{j}\times\hat{i}=\hat{k}\\ \hat{k}\times\hat{i}=-\hat{i}\times\hat{k}=\hat{j}\\ \hat{j}\times\hat{k}=-\hat{k}\times\hat{j}=\hat{i}\end{array}\right\}$$ Right-Handed Coordinate System [Definition]

$$\vec{A}\times\vec{B}=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$

$$=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}$$

$$+(A_xB_y-A_yB_x)\hat{k}$$

[Result following multiplication term-by-term and elimination of many vanishing terms]

$$\vec{A}\times\vec{B}\equiv\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ A_x&A_y&A_z\\ B_x&B_y&B_z\end{vmatrix}$$

$\parallel$ – vanish !
$\perp$ – survive !

Determinants

a) **Order-2**

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Example: $$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3(5) - 4(-2) = 23$$

b) **Order-3**

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example:

$$\begin{vmatrix} 3 & 2 & -1 \\ 4 & 3 & 3 \\ -2 & 7 & 1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 3 \\ 7 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ -2 & 7 \end{vmatrix}$$

$$= 3\left[3(1) - 7(3)\right] - 2\left[4(1) - (-2)(3)\right] + (-1)\left[4(7) - (-2)(3)\right]$$

$$= -54 - 20 - 34$$

$$= \underline{-108}$$

Applications of the cross-product will include angular momentum, $\vec{L}$, and torque, $\vec{\tau}$.

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F}$$

Example - Cross Product

$$\vec{A} = 3\hat{i} + 7\hat{j} - \hat{k}$$
$$\vec{B} = \hat{i} - \hat{j}$$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ +3 & +7 & -1 \\ +1 & -1 & 0 \end{vmatrix}$$

$$= \hat{i}\left[7(0) - (-1)(-1)\right] - \hat{j}\left[3(0) - (1)(-1)\right] + \hat{k}\left[3(-1) - (1)(7)\right]$$

$$= -\hat{i} - \hat{j} - 10\hat{k}$$

$$\vec{C}\cdot\vec{A} = (-\hat{i} - \hat{j} - 10\hat{k})\cdot(3\hat{i} + 7\hat{j} - \hat{k}) = -3 - 7 + 10 = 0$$

$$\vec{C}\cdot\vec{B} = (-\hat{i} - \hat{j} - 10\hat{k})\cdot(\hat{i} - \hat{j}) = -1 + 1 = 0$$

$\vec{C}$ is $\perp$ to both $\vec{A}$ and $\vec{B}$

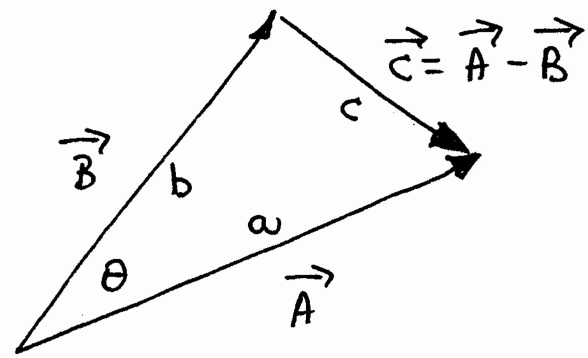
$C = AB \sin\theta$ [magnitudes]

$$\sin\theta = \frac{C}{AB} = \frac{\sqrt{(-1)^2 + (-1)^2 + (-10)^2}}{\sqrt{3^2 + 7^2 + (-1)^2}\,\sqrt{1^2 + (-1)^2}} = \frac{\sqrt{102}}{\sqrt{59}\,\sqrt{2}}$$

$$= \sqrt{\frac{102}{118}}$$

$$\theta = \sin^{-1}\sqrt{} = 70.8^\circ$$

Example



$$\vec{C} = \vec{A} - \vec{B} \qquad [\text{Form } \Delta]$$

$$\vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$

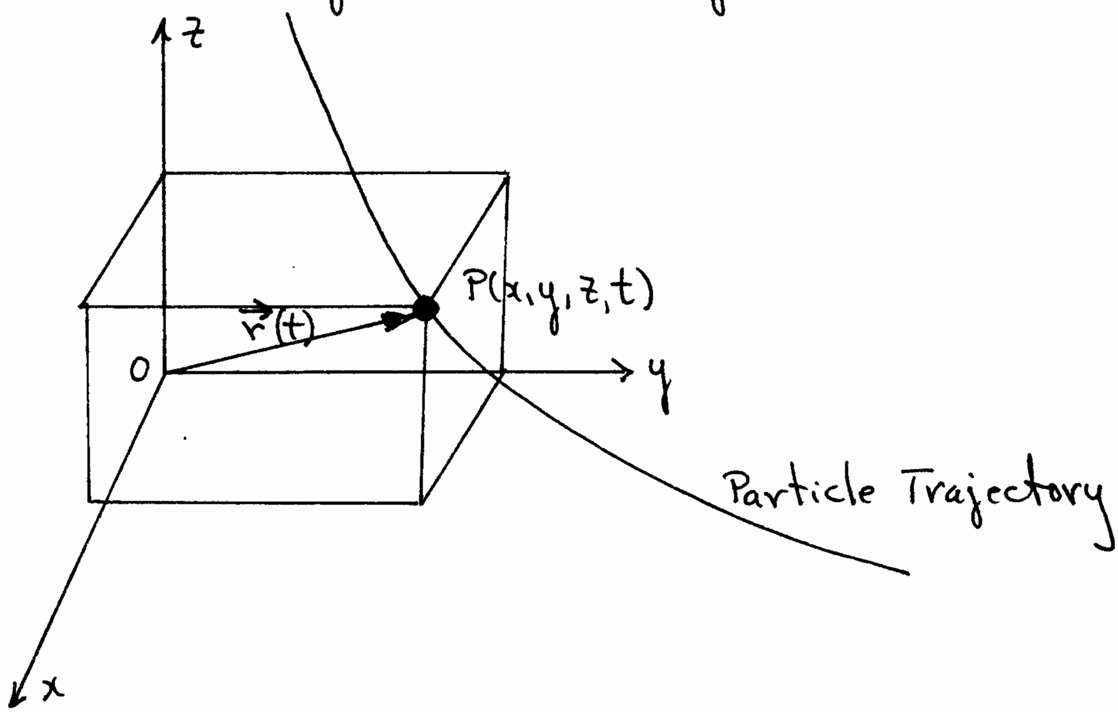
$$= \vec{A}\cdot\vec{A} - 2\,\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$$

$$c^2 = a^2 - 2ab\cos\theta + b^2$$

Law of Cosines

Kinematics in Three Dimensions

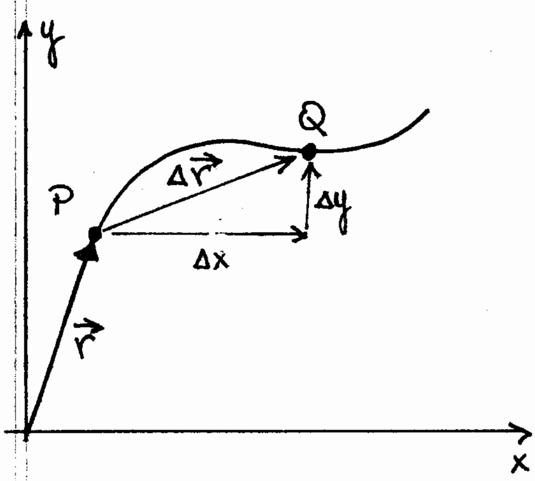
- extension of our description of 1-dimensional motion
- 3-dimensional motion is basically 3 - one dimensional motions occuring simoltaneously.



$\vec{r}(t) \equiv$ instantaneous position vector of the particle.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Average velocity



Consider a particle in a plane moving from P to Q.
Position vector at P is $\vec{r}$.
The change in position vector is $\vec{\Delta r}$

$$\vec{\Delta r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

Let Δt be the time interval for the motion from P to Q. The <u>average velocity</u> of the particle is then defined as the vector quantity equal to the displacement divided by the time interval:

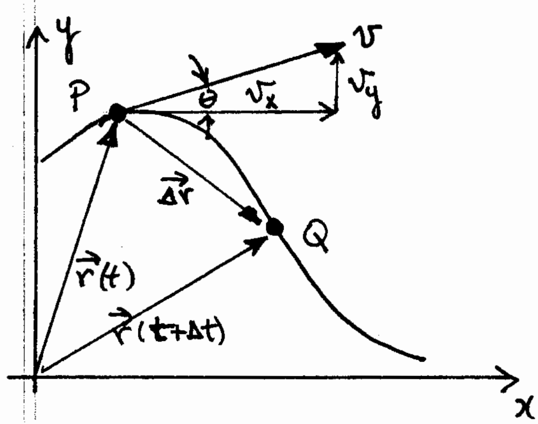
$$\vec{\bar{v}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k} \qquad [3\text{-D}]$$

- direction of vector $\vec{\Delta r}$
- magnitude $\Delta r/\Delta t$

Let $\bar{v}_x = \frac{\Delta x}{\Delta t}$, $\bar{v}_y = \frac{\Delta y}{\Delta t}$, $\bar{v}_z = \frac{\Delta z}{\Delta t}$

$$\bar{v} = \bar{v}_x\,\hat{i} + \bar{v}_y\,\hat{j} + \bar{v}_z\,\hat{k}$$

<u>Instantaneous velocity</u>, v, at the point P is defined in <u>magnitude</u> and <u>direction</u> as the limit approached by the average velocity when point Q is taken to be closer and closer to P (as $\Delta t \to 0$).



$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- direction of $\vec{v}$ is tangent to path of particle at P.
- equal to time rate of change of position vector.
- describes motion at a particular point and particular instant of time

$$\vec{v} = \frac{d}{dt}\left[x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right]$$

$$= \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k} \qquad \text{since } \frac{d\hat{i}}{dt} = 0, \text{ etc.}$$

$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k}$$

Magnitude of $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ [rect. coord.]

$$\tan\theta = v_y / v_x \qquad [2\text{-D}]$$

Represent $\vec{v}$: components, or
magnitude and direction

Speed

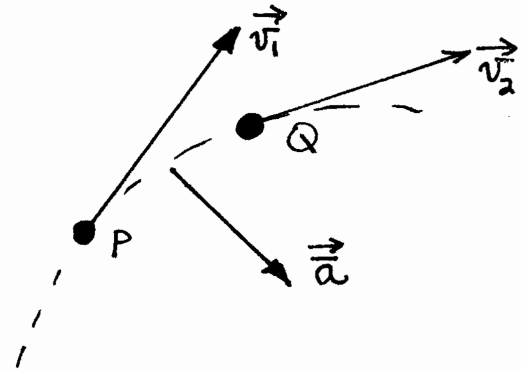
$$[\text{speed}] = v = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- instantaneous speed is magnitude of velocity vector.
- velocity has direction and magnitude.

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time}}$$

Acceleration

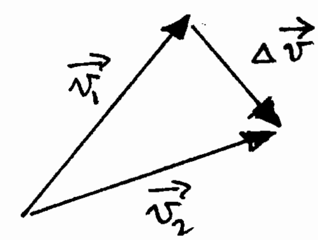
The average and instantaneous accelerations in 3-dimensions are straightforward generalizations of one-dimensional motion.



$\vec{v}_1$ and $\vec{v}_2$ are velocities for particle moving in a curved path at P and Q.

- different magnitudes
- different direction

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$



The average acceleration, $\bar{\vec{a}}$, of the particle as it moves from P to Q is defined as the vector change in velocity, $\Delta\vec{v}$, divided by the time interval Δt.

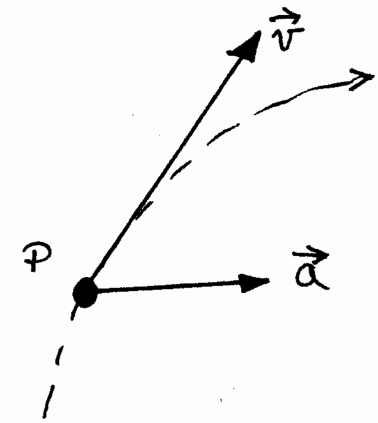
$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} + \frac{\Delta v_z}{\Delta t}\hat{k}$$

$$= \bar{a}_x\hat{i} + \bar{a}_y\hat{j} + \bar{a}_z\hat{k}$$

$\bar{\vec{a}} \neq 0$ if velocity changes direction between P+Q or if velocity changes in magnitude.

The instantaneous acceleration, $\vec{a}$, at point P is defined in magnitude and direction as the limit approached by the average acceleration when point Q approaches point P and Δv and Δt both approach zero



$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$= a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

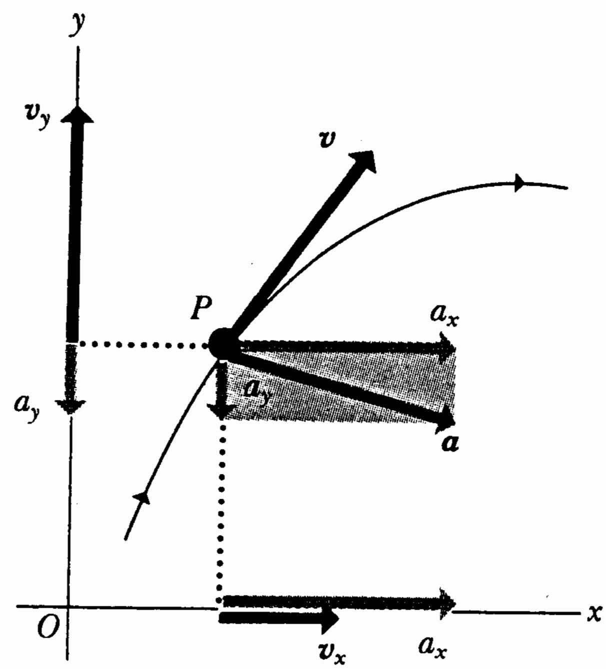
$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

- Equal to time rate of change of velocity
- $\neq 0$ if velocity changes in magnitude or direction
- It does not have same direction as velocity vector.
- Acceleration vector lies on concave side of curved path.

- average acceleration refers to finite time interval during which velocity changes.
- instantaneous acceleration is the rate of change of velocity at a specific point at a specific time.



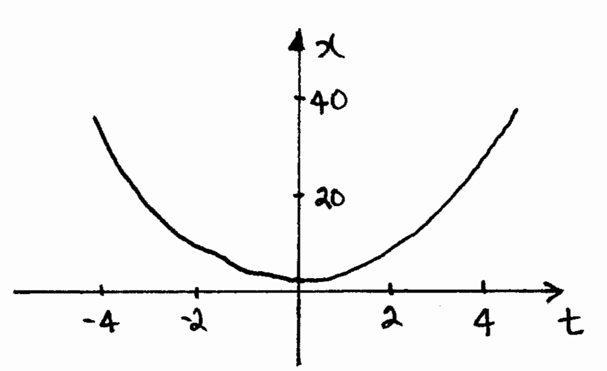
The acceleration ***a*** is resolved into its components a_x and a_y.

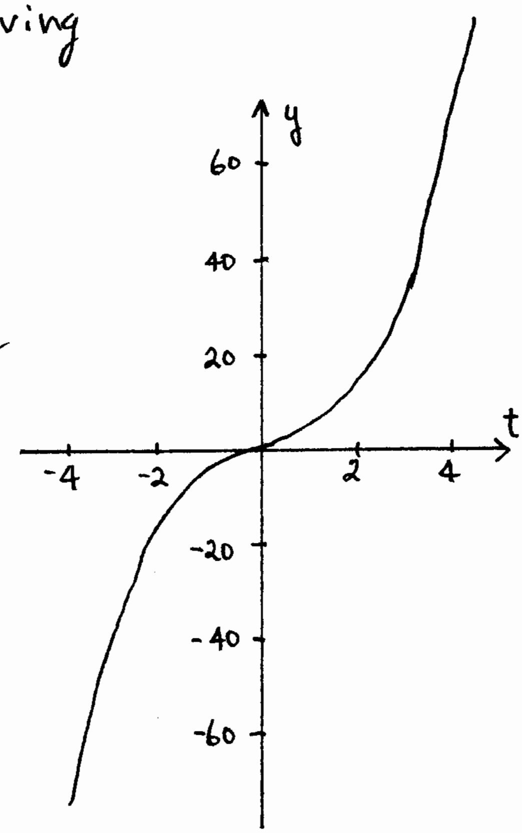
Example

The coordinates of a particle moving in the xy-plane are given by

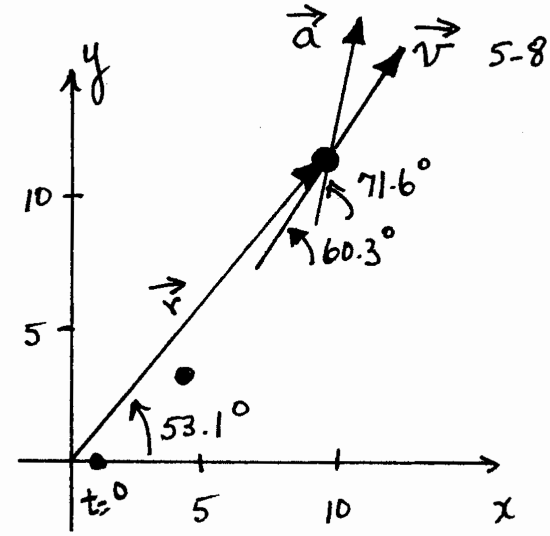
$$x = 1 + 2t^2 \qquad (m)$$
$$y = 2t + t^3 \qquad (m)$$

Find the particle's position, velocity and acceleration at time $t = 2s$.





Position



$$x = 1 + 2(2)^2 = 9\,m$$
$$y = 2(2) + 1(2)^3 = 12\,m$$

$$\vec{r} = 9\hat{i} + 12\hat{j}$$

Distance from origin

$$r = \sqrt{x^2 + y^2} = \sqrt{9^2 + 12^2} = 15\,m$$
$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{12}{9} = 53.1°$$

Velocity

$$v_x = \frac{dx}{dt} = 4t \quad (m/s) \qquad v_y = \frac{dy}{dt} = 2 + 3t^2 \quad (m/s)$$

At $t = 2s$ $\quad v_x(2) = 8\ m/s \quad v_y(2) = 14\ m/s$

$$\vec{v}(t=2) = 8\hat{i} + 14\hat{j}$$

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{8^2 + 14^2} = 16.1\,m/s$$
$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{14}{8} = 60.3°$$

Acceleration

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 4 \quad (m/s^2) \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 6t \quad (m/s^2)$$

At $t = 2s$ $\quad a_x = 4\ m/s^2 \quad a_y = 12\ m/s^2$

$$\vec{a} = 4\hat{i} + 12\hat{j}$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2 + 12^2} = 12.6\,m/s^2$$
$$\theta_a = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{12}{4} = 71.6°$$

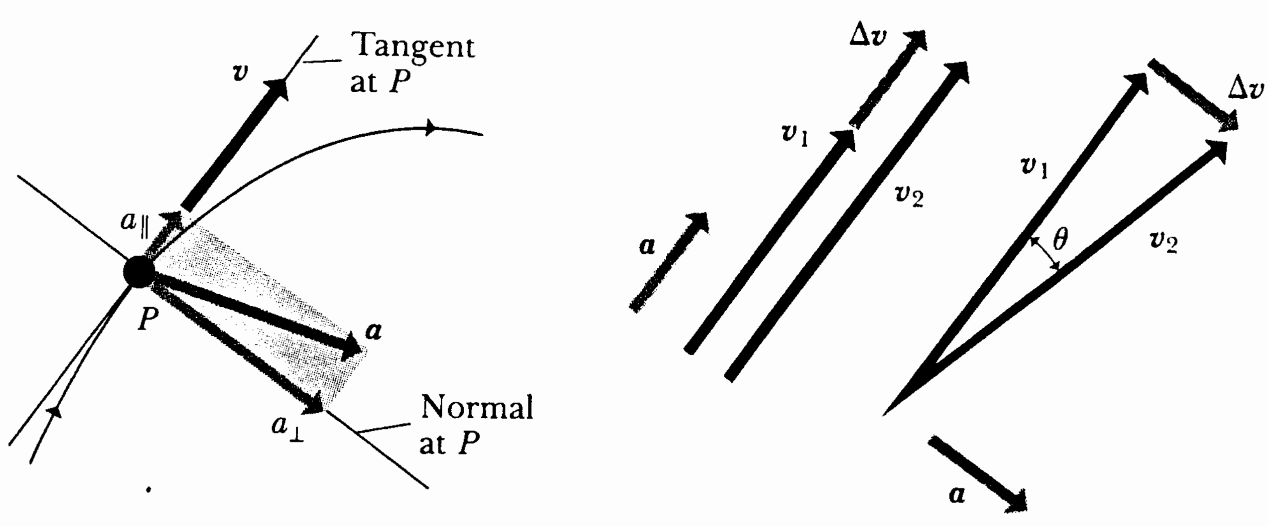
[Direction of $\vec{a} \neq \vec{v}$ and not tangent to path]

Acceleration: $a_\perp$ and $a_\parallel$

- Instructive to represent the acceleration of a particle moving along a curved path in terms of rectangular components:
  - $a_\perp$ – normal to path
    - associated with change in direction of $\vec{v}$
  - $a_\parallel$ – parallel to path
    - associated with change in magnitude of $\vec{v}$

$\vec{a}$ parallel to $\vec{v}$: Change in $\vec{v}_1$ during a small amount of time Δt is a vector $\Delta\vec{v}$ with same direction as $\vec{a}$ and same direction as $\vec{v}_1$. Velocity $\vec{v}_2$ at the end of Δt, is $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, same direction as $\vec{v}_1$ but somewhat greater in magnitude.

$\vec{a}$ perpindicular to $\vec{v}$: In time interval Δt the change $\Delta\vec{v}$ is $\perp$ to $\vec{v}_1$. $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$ but $\vec{v}_1$ and $\vec{v}_2$ differ in direction. As $\Delta t \to 0$, $\Delta\vec{v}$ is $\perp$ to $\vec{v}_1$ and $\vec{v}_2$. $\vec{v}_1$ and $\vec{v}_2$ have same magnitude.



$$\vec{a} = \frac{d\vec{v}}{dt}$$

$d\vec{v} = \vec{a}\, dt$ change in velocity is in same direction as acceleration vector.

Using $\vec{a}_{\parallel}$ and $\vec{a}_{\perp}$ one easily sees that if

$\vec{a} \cdot \vec{v} > 0$ velocity increases

$\vec{a} \cdot \vec{v} < 0$ velocity decreases.

$\vec{a} \cdot \vec{v} \equiv 0$ velocity magnitude remains constant
velocity changes direction
$\Rightarrow$ circular motion !!

Motion at Constant Acceleration

We had for the average acceleration

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

If initial velocity is $\vec{v}_0$, then the velocity after time t is

$$\vec{v} = \vec{v}_0 + \vec{a} t \qquad (1)$$

for constant acceleration.

The x, y, and z components are:

$$\begin{aligned} v_x &= v_{0x} + a_x t \\ v_y &= v_{0y} + a_y t \\ v_z &= v_{0z} + a_z t \end{aligned} \qquad (2)$$

Using arguments as in the one-dimensional case the position vector becomes

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2 \qquad (3)$$

In components:

$$\begin{aligned} x(t) &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \\ y(t) &= y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \\ z(t) &= z_0 + v_{0z} t + \frac{1}{2} a_z t^2 \end{aligned} \qquad (4)$$

In equations ② and ④ the various components of the motion proceed independent of each other. Time is common ~~tell~~ to all motions.

x-velocity affected by x-acceleration

x-position affected by initial x-velocity and x-acceleration