Constant $\vec{a}$ Motion: 3D

$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$

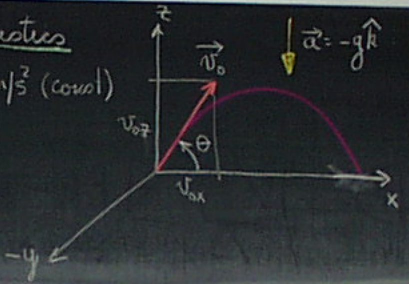
$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$

$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$

$z(t) = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$

Motion of Projectiles: Ballistics

- Short Range $g = 9.81\,m/s^2$ (const)
- No air resistance
- No earth rotation
- Constant $\vec{a}$ motion



Assume:

①

- Motion in xz-plane only
- z-axis is vertical
- x-axis horizontal

$a_x = 0$ $\quad v_{0y} = 0$

$a_y = 0$ $\quad y_0 = 0$

$a_z = -g\hat{k}$ $\quad g = 9.81\,m/s^2$

Eg's of Motion:

②

$x(t) = x_0 + v_{0x}t$ ①

$y(t) = 0$ ②

$z(t) = z_0 + v_{0z}t - \frac{1}{2}gt^2$ ③

$v_x(t) = v_{0x}$ ④

$v_z(t) = v_{0z} - gt$ ⑤

$v_y(t) = 0$ ⑥

$x(t), y(t), z(t) \Rightarrow$ particle traj. vs t.

Motions are decoupled (Exp)

2D Motion - General

1D Special Case

What kind of trajectory is ballistic motion?

From ① $x(t) = x_0 + v_{0x}t$

$t = \frac{x - x_0}{v_{0x}}$

$z(t) = z_0 + v_{0z}\left(\frac{x - x_0}{v_{0x}}\right) - \frac{1}{2}g\left(\frac{x - x_0}{v_{0x}}\right)^2$

$z(t) = A + Bx^2 + Cx^2$
A, B, C Constants } Parabola.

Ballistic Motion

Missile Ball

Bullet Bomb

Calculate:

- Max Height
- Time-of-Flight
- Range

Assume:

$z(0) = 0$, $x(0) = 0$ } Origin at $t = 0$

$v_z(0) = v_{z0}$, $v_x(0) = v_{x0}$ } Initial Velocity Components

Lecture 6, Blackboard #1

Eq's of Motion:

$x(t) = x_0 + v_{0x}\, t$ ①

$y(t) = 0$ ②

$z(t) = z_0 + v_{0z} t - \frac{1}{2} g t^2$ ③

$v_x(t) = v_{0x}$ ④

$v_z(t) = v_{0z} - g t$ ⑤

$v_y(t) = 0$ ⑥

$x(t), y(t), z(t) \Rightarrow$ particle traj. vs t.

Motions are decoupled (Exp)

2D Motion - General

1D Special Case

What kind of trajectory is ballistic motion?

From ① $x(t) = x_0 + v_{0x} t$

$$t = \frac{x - x_0}{v_{0x}}$$

$$z(t) = z_0 + v_{0z}\left(\frac{x - x_0}{v_{0x}}\right) - \frac{1}{2} g \left(\frac{x - x_0}{v_{0x}}\right)^2$$

$$\left.\begin{array}{l} z(t) = A + Bx^2 + Cx^2 \\ A, B, C \text{ Constants} \end{array}\right\} \text{Parabola.}$$

Ballistic Motion

Missile Ball

Bullet Bomb

Calculate:

Max Height

Time-of-Flight

Range

②

Assume:

$$\left.\begin{array}{l} z(0) = 0 \\ x(0) = 0 \end{array}\right\} \text{Origin at } t = 0$$

$$\left.\begin{array}{l} v_z(0) = v_{z0} \\ v_x(0) = v_{x0} \end{array}\right\} \text{Initial Velocity Components}$$

③

Motion in xz-plane only!

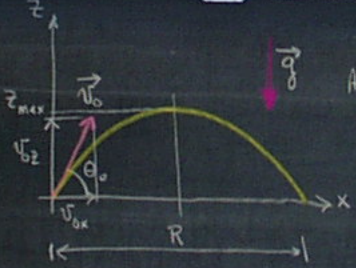
At max height: $v_z(t_{max}) = 0$

$0 = v_{0z} - g t_{max}$

$t_{max} = \frac{v_{0z}}{g}$ Time-to-max.

$z(t_{max}) = v_{0z} t_{max} - \frac{1}{2} g t_{max}^2$

$$= v_{0z} \frac{v_{0z}}{g} - \frac{1}{2} g \frac{v_{0z}^2}{g^2} = \frac{v_{0z}^2}{2g}$$



$\vec{v}(t) = v_{0x} \hat{i} + v_{0z} \hat{k}$

Initial Velocity

What is Range, R?

At impact $\Rightarrow z = 0$!

$0 = v_{0z} t_F - \frac{1}{2} g t_F^2$

$t_F \Rightarrow$ Flight Time

Solve:

$t_F = 0$ (Start)

$t_F = \frac{2 v_{0z}}{g}$ (Impact)

$= 2 t_{max}$

x-motion:

$R = v_{0x} t_F$

$= \frac{2 v_{0x} v_{0z}}{g}$ (m)

Velocity:

$v(t) = v_x(t) \hat{i} + v_z(t) \hat{k}$

$|v| = \sqrt{v_x^2 + v_z^2}$

$\tan\theta = \frac{v_z(t)}{v_x(t)}$ Direction

What is $\vec{v}$ at $z = z_{max}$?

$v_x(t) = v_{0x}$ constant of motion.

$v_z(t = t_{max}) = v_{0z} - g \frac{v_{0z}}{g} = 0$

$\vec{v}(t_{max}) = v_{0x} \hat{i}$

Trajectory is Horizontal.

What is $\vec{v}$ at $x = R$?

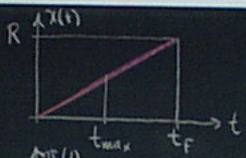
$v_x(t) = v_{0x}$

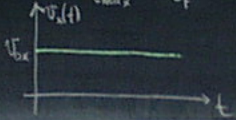
$v_z\left(t = \frac{2v_{0z}}{g}\right) = v_{0z} - g\left(\frac{2v_{0z}}{g}\right) = -v_{0z}$

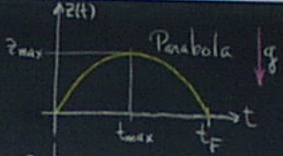
$v = v_{0x}\hat{i} - v_{0z}\hat{k}$

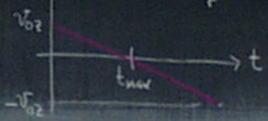
$|v| \equiv$ same as at origin at $t=0$

$\theta = -\theta_0$ Direction!









4

Projectile

- Launch angle θ
- Initial Speed v_0

$v_{0x} = v_0 \cos\theta$

$v_{0y} = v_0 \sin\theta$

$x(t) = (v_0\cos\theta)t + x_0$

$z(t) = z_0 + (v_0\sin\theta)t - \frac{1}{2}gt^2$

$v_x(t) = v_0\cos\theta$ (Const)

$v_z(t) = v_0\sin\theta - gt$ (Varies)

$z_{max} = \frac{v_0^2\sin^2\theta}{2g}$ $\quad z_0 = 0$

$t_F = \frac{2v_0\sin\theta}{g}$

$x_{max} = R = \frac{2v_0^2\sin\theta\cos\theta}{g}$

$= \frac{v_0^2\sin 2\theta}{g}$ $(z_0 = 0)$

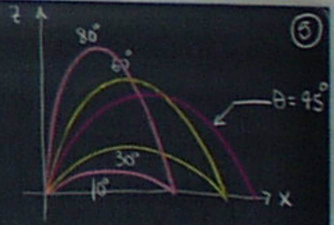
Max R: ?

$\sin 2\theta = 1$

$2\theta = 90^\circ$

$\theta = 45^\circ$

$z_{max}(\theta = 90^\circ) = \frac{v_0^2}{2g}$



5

Projectile

- Launch angle θ
- Initial Speed v_0

$v_{0x} = v_0 \cos\theta$

$v_{0y} = v_0 \sin\theta$

$x(t) = (v_0 \cos\theta)\, t + x_0$

$z(t) = z_0 + (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$

$v_x(t) = v_0 \cos\theta$ (Const)

$v_z(t) = v_0 \sin\theta - gt$ (Varies)

$z_{max} = \dfrac{v_0^2 \sin^2\theta}{2g}$ $\quad z_0 = 0$

$t_F = \dfrac{2 v_0 \sin\theta}{g}$

$x_{max} = R = \dfrac{2 v_0^2 \sin\theta \cos\theta}{g}$

$= \dfrac{v_0^2 \sin 2\theta}{g}$ $\quad (z_0 = 0)$

Max R: ?

$\sin 2\theta = 1$

$2\theta = 90^\circ$

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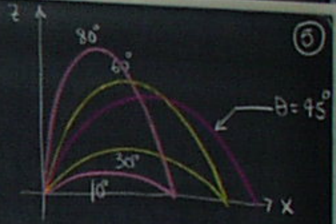
$z_{max}(\theta = 90^\circ) = \dfrac{v_0^2}{2g}$



(5)

$\theta = 0^\circ$ $\quad z_{max} = \sim 3.0\,\text{m} = v_0^2/2g$

θ	$\sin 2\theta$	Range-1	Range-2
× 15	.500		
× 20	.643		
30	.866		
× 45	1.000		
× 70	.643		

$R = \dfrac{v_0^2}{g} \sin 2\theta$

$= 2 z_{max} \sin 2\theta$

(6)

Lecture 6, Blackboard #4

Ballistics Problem Strategy

$x(t) = x_0 + (v_0 \cos\theta)\, t$

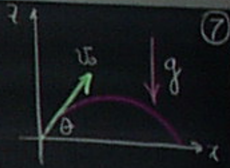
$z(t) = z_0 + (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$

$v_x(t) = v_0 \cos\theta$

$v_z(t) = v_0 \sin\theta - g t$

1. Define coord axes: Origin/Scale
2. List knowns and unknowns
3. Prose → Symbols
   - When → Time
   - Where → Position
   - Velocity
4. Have a picture of the trajectory
   - Parabola
5. Highest point $v_z = 0$
6. Range: $z = 0$ **OR** some final value.
7. Study answer
   - Does it make sense?
   - Sign? Units?

7



Example Falling Apple

Arrow fired at same time apple falls

How to aim to strike apple?

$\left.\begin{array}{l} x_D(t) \\ z_D(t) \end{array}\right\}$ Traj. of Dart

$\left.\begin{array}{l} x_A(t) \\ z_A(t) \end{array}\right\}$ Traj of Apple

$\left.\begin{array}{l} z_D(T) = z_A(T) \\ x_D(T) = x_A(T) \end{array}\right\}$ Cond for Collision

Apple: $x_A(t) = L$

$z_A(t) = H - \frac{1}{2} g t^2$

Dart: $x_D(t) = v_{0x} t$

$z_D(t) = v_{0z} t - \frac{g t^2}{2}$

At $t = T$ $\quad L = v_{0x} T$ (x-motion)

$\therefore T = L / v_{0x}$ Time to Impact.

$v_{0z} T - \frac{1}{2} g T^2 = H - \frac{1}{2} g T^2$

$\frac{v_{0z} L}{v_{0x}} = H \qquad \frac{v_{0z}}{v_{0x}} = \frac{H}{L} \qquad v_{0x} = v_0 \cos\theta, \quad v_{0z} = v_0 \sin\theta$

$\frac{H}{L} = \frac{\sin\theta}{\cos\theta} = \tan\theta \qquad \theta \Rightarrow$ Points directly at apple

Independent of v_0! $\quad v_0 > v_{0\,min}$

8