Center Of Mass

Particles: $\vec{r}_{cm} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots}$

Solid Object: $\vec{r}_{cm} = \dfrac{1}{M}\int \rho\,\vec{r}\,dv$

$x_{cm} = \dfrac{1}{M}\int \rho\, x\, dv$

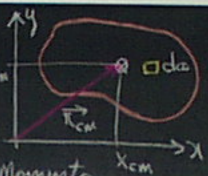
y_{cm}, z_{cm}: Same

Areas/Plates

$\sigma(x,y)$: Areal Density kg/m^2

$dm_i = \sigma\, da_i$

$$\left.\begin{aligned} x_{cm} &= \frac{1}{M}\int \sigma\, x\, da \\ y_{cm} &= \frac{1}{M}\int \sigma\, y\, da \end{aligned}\right\} \text{1st Moments of Area.}$$



Example: Right Circular Cone

By symmetry: $x_{cm} = 0$; $y_{cm} = 0$

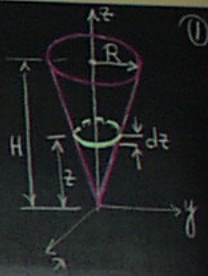
Cone centered on z-axis.

Cut out discs: thickness dz

$dm = \rho\, dv = \rho 2\pi r^2 dz$

r: radius of disc at loc. z

ρ = uniform density.



①

Geometry: $\dfrac{r}{R} = \dfrac{z}{H}$ $\therefore r = z\dfrac{R}{H}$

$M = \rho \times \text{Volume} = \dfrac{\rho\pi R^2 H}{3}$

$z_{cm} = \dfrac{1}{M}\int z\, dm = \dfrac{1}{M}\int_{z=0}^{H} \rho z \pi r^2 dz$

$z_{cm} = \dfrac{1}{M}\int_0^H \dfrac{\rho\pi R^2}{H^2} z^3 dz$

$z_{cm} = \dfrac{\rho\pi R^2}{MH^2}\left.\dfrac{z^4}{4}\right|_0^H = \dfrac{\rho\pi R^2 H^2}{4M} = \dfrac{3}{4}H \Rightarrow \dfrac{H}{4}$ From Base!

Example: CM Disk + Hole

Density: σ g/cm^2

$y_{cm} = 0$ Symmetry

$(x_{cm})_{NoHole} = 0$

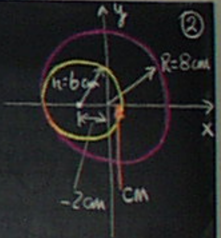
Two Objects: Disk NoHole

Hole: Neg Mass.

$$\left. x_{cm}\right|_{Hole} = \frac{(\pi R^2\sigma)(0) - (\pi r^2\sigma)(-2cm)}{(\pi R^2 - \pi r^2)\sigma}$$

$$= \left(\frac{r^2}{R^2 - r^2}\right)(-2cm)$$

$$= \frac{6^2}{8^2 - 6^2}(-2) = 2.6cm$$



②

moving with velocity of v_{cm}

Lecture 18, Blackboard #1

Geometry: $\frac{r}{R} = \frac{z}{H}$ ∴ $r = z\frac{R}{H}$

$M = \rho \times \text{Volume} = \frac{\rho \pi R^2 H}{3}$

$$z_{cm} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int_{z=0}^{H} \rho z \pi r^2\, dz$$

$$z_{cm} = \frac{1}{M}\int_0^H \frac{\rho\pi R^2}{H^2} z^3\, dz$$

$$z_{cm} = \frac{\rho\pi R^2}{MH^2}\frac{z^4}{4}\Big|_0^H = \frac{\rho\pi R^2H^2}{4M} = \frac{3}{4}H \Rightarrow \frac{H}{4} \text{ From Base!}$$

Example: CM Disk + Hole

Density: σ g/cm^2

$y_{cm} = 0$ Symmetry

$(X_{cm})_{NoHole} = 0$

Two Objects: Disk NoHole

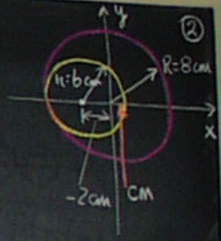
Hole: Neg Mass.

$$X_{cm}\Big|_{Hole} = \frac{(\pi R^2\sigma)(0) - (\pi r^2\sigma)(-2cm)}{(\pi R^2 - \pi r^2)\sigma}$$

$$= \left(\frac{r^2}{R^2-r^2}\right)(-2cm)$$

$$= \frac{6^2}{8^2-6^2}(-2) = 2.6cm$$

②



Motion of CM

$$\vec v_{cm} = \frac{d\vec r_{cm}}{dt} = \frac{1}{M}\left[m_1\frac{d\vec r_1}{dt} + m_2\frac{d\vec r_2}{dt} + \dots m_n\frac{d\vec r_n}{dt}\right]$$

Assume M = const.

$$\vec v_{cm} = \frac{1}{M}\left[m_1\vec v_1 + m_2\vec v_2 + \dots m_n\vec v_n\right]$$

$= \vec P/M$ Total Linear Momentum / Mass

∴ $\vec P = M\vec v_{cm}$ Total $\vec P$ is the same as of a single particle of mass M moving with velocity of $\vec v_{cm}$

$$\vec a_{cm} = \frac{d\vec v_{cm}}{dt} = \frac{1}{M}\sum m_i \frac{d\vec v_i}{dt} = \frac{1}{M}\sum m_i\vec a_i$$

∴ $M\vec a_{cm} = \sum \vec F_i$

$\vec F_i = \vec F_{i,INT} + \vec F_{i,ext}$ $\quad \sum\vec F_{i,INT} = 0$

∴ $\boxed{\sum F_{i,EXT} = M\vec a_{cm} = \frac{d\vec P}{dt}}$

CM moves like imaginary particle M, with resultant force.

If $\sum F_{i,EXT} = 0$ $\quad \frac{d\vec P}{dt} = 0$ $\quad \vec P = M a_{cm}$ = Constant

Example: Exploding Shell

③

$\sum F_{Ext,\,x} = 0$

$\sum F_{Ext,\,y} \neq 0$

Lecture 18, Blackboard #2

Example CM Motion

④

$\sum \vec F_{ext} = M\vec a_{cm}$

$\vec F + \cancel{\vec N_1} + \cancel{\vec N_2} + \cancel{\vec w_1} + \cancel{\vec w_2} + \cancel{\vec N_3} + \cancel{\vec N_4} = M\vec a_{cm}$

$12\hat i = 6\vec a_{cm}$

$\vec a_{cm} = 2\hat i$

$\vec a_{m_1} = 2\hat i \quad \vec a_{m_2} = 2\hat i$

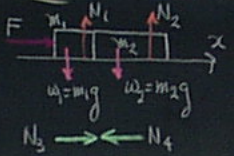
$\vec N_1 = -\vec w_1$

$\vec N_2 = -\vec w_2$

$\vec N_3 = -\vec N_4$

$\vec F = 12\hat i$

$M = m_1 + m_2 = 6\,kg$



$m_1 = 2\,kg \quad L_1 = 20\,cm$

$m_2 = 4\,kg \quad L_2 = 40\,cm$

$\vec a_1 \equiv 0 \; ; \; \vec v_1 = 0 \; ; \; \vec r_1 = 0$

$\vec a_2 = \frac{12\hat i}{4} = 3\hat i$

$\vec v_2 = 3t\,\hat i$

$\vec r_2 = (0.3m + 1.5t^2)\,\hat i$

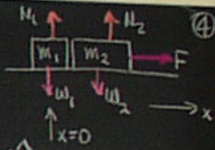
$M\vec r_{cm} = \sum m_i \vec r_i$

$M\vec v_{cm} = \sum m_i \vec v_i$

$M\vec a_{cm} = \sum m_i \vec a_i$

$\vec a_{cm} = \frac{1}{6}\left[0 + 4\times 3\hat i\right] = 2\hat i$

Exactly the same as before!!!



Energy of a System of Particles:

⑤

Momentum $\vec P = M\vec v_{cm}$

Kinetic Energy $K = \frac12 M v_{cm}^2$??

Total KE: $K = \sum_i K_i$

$K = \sum_i \frac12 m v_i^2 = \frac12 m_1 v_1^2 + \frac12 m_2 v_2^2 + \cdots \frac12 m_n v_n^2$

Rewrite K in terms of $\vec v_{cm}$ and $\vec u_i$: What does it look like?

$\vec v_1$: velocity in Lab Frame.

$\vec v_{cm}$: velocity of CM rel. to Lab F.

$\vec u_1$: velocity of m_1 in CM Frame.

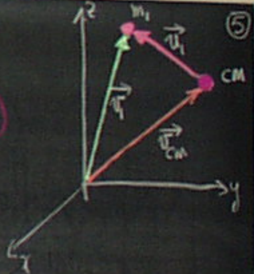
$$\left.\begin{aligned} \vec u_1 &= \vec v_1 - \vec v_{cm} \\ \vec u_n &= \vec v_n - \vec v_{cm} \\ \vec v_1 &= \vec u_1 + \vec v_{cm} \end{aligned}\right\} \text{Galilean Transformations}$$

$K = \frac12 m_1(\vec u_1 + \vec v_{cm})^2 + \cdots \frac12 m_n(\vec u_n + \vec v_{cm})^2$

$= \frac12 m_1 u_1^2 + \frac12 m_2 u_2^2 + \cdots \frac12 m_n u_n^2$ T1

$+ (m_1\vec u_1 + m_2\vec u_2 + \cdots m_n\vec u_n)\cdot\vec v_{cm}$ + T2

$+ \frac12(m_1 + m_2 + \cdots m_n)\, v_{cm}^2$ T3



Lecture 18, Blackboard #3

Energy of a System of Particles:

Momentum $\vec{P} = M\vec{v}_{cm}$

Kinetic Energy $K = \frac{1}{2}M v_{cm}^2$??

Total KE: $K = \sum_i K_i$

$K = \sum_i \frac{1}{2} m v_i^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \dots \frac{1}{2}m_n v_n^2$

Rewrite K in terms of $\vec{v}_{cm}$ and $\vec{u}_i$: What does it look like?

$\vec{v}_1$: velocity in Lab Frame.

$\vec{v}_{cm}$: velocity of CM rel. to Lab F.

$\vec{u}_1$: velocity of m_1 in CM Frame

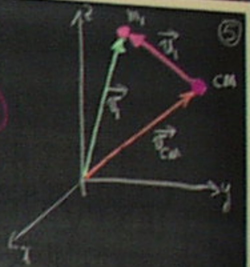
$$\left.\begin{aligned} \vec{u}_1 &= \vec{v}_1 - \vec{v}_{cm} \\ u_n &= \vec{v}_n - \vec{v}_{cm} \\ \vec{v}_1 &= \vec{u}_1 + \vec{v}_{cm} \end{aligned}\right\} \text{Galilean Transformations}$$

$$K = \frac{1}{2}m_1(\vec{u}_1 + \vec{v}_{cm})^2 + \dots \frac{1}{2}m_n(\vec{u}_n + \vec{v}_{cm})^2$$

$$= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 + \dots \frac{1}{2}m_n u_n^2 \quad \text{T1}$$

$$+ (m_1\vec{u}_1 + m_2\vec{u}_2 + \dots m_n\vec{u}_n)\cdot\vec{v}_{cm} + \quad \text{T2}$$

$$+ \frac{1}{2}(m_1 + m_2 + \dots m_n)\, v_{cm}^2 \quad \text{T3}$$



5

First Term: T1

$K_{INT} = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 + \dots \frac{1}{2}m_n u_n^2$

$\Longrightarrow$ Internal Kinetic Energy

- Motions
- Rotations
- Heat.

Third Term: T3

$= \frac{1}{2}M v_{cm}^2$ KE due to motion of CM of total mass M of all particles.

Second Term: T2

$$[m_1\vec{u}_1 + \dots]\cdot\vec{v}_{cm} = [m_1(\vec{v}_1 - \vec{v}_{cm}) + m_2(\vec{v}_2 - \vec{v}_{cm}) + \dots]\cdot\vec{v}_{cm}$$

$$= [(m_1\vec{v}_1 + m_2\vec{v}_2 + \dots) - (m_1 + m_2 + \dots)\vec{v}_{cm}]\cdot\vec{v}_{cm}$$

$$= [M\vec{v}_{cm} - M\vec{v}_{cm}]\cdot\vec{v}_{cm} \equiv 0.$$

Summary:

$$K = \underbrace{K_{INT}}_{\text{Int KE}} + \underbrace{\frac{1}{2}M v_{cm}^2}_{\text{KE of CM Motion}}$$

Potential Energy / CM:

U = Function of position of all the particles.

$\boxed{\text{Total } E = \text{Total } K + \text{Total } U}$

Gravitational PE – Extended Body

$U = (m_1 z_1 + m_2 z_2 + \dots m_n z_n)\, g$

$= M z_{cm} g$ Behaves like mass M at CM.

6

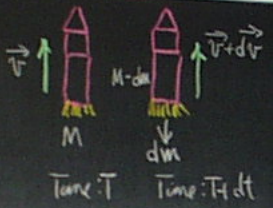
Rockets/Variable Mass Problems.

$\vec F_{ext} = \frac{d\vec P}{dt} = \frac{d(M\vec v)}{dt}$ Eq. of Motion

What happens when mass M varies with time?

Rocket propelled forward by action of ejection of gases $\Rightarrow$ acceleration!

Rocket mass decreases with time: $\frac{dM}{dt}<0$



Rocket Mass = $M(t)$

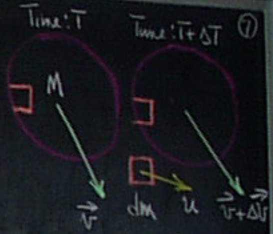
Ejects mass Δm in time Δt.

$\vec P_i = M\vec v$ (t)

$\vec P_f = (M-\Delta m)(\vec v + \Delta \vec v)$ $(t+\Delta t)$

$\vec v$: velocity of rocket in inertial frame (earth)

$\vec u$: velocity of Δm in same inertial frame.



⑦

$\Delta\vec P = \vec P_f - \vec P_i = \cancel{M\vec v} + M\Delta\vec v - \Delta m\vec v - \cancel{\Delta m\Delta\vec v}$
$+ \vec u\Delta m - \cancel{M\vec v}$

$\Delta\vec P = M\Delta\vec v + (\vec u - \vec v)\Delta m$

$\Delta m = -\Delta M$ Let $\Delta t \to 0$

$\vec F_{ext} = \frac{d\vec P}{dt} = M\frac{d\vec v}{dt} - (\vec u - \vec v)\frac{dM}{dt}$

$\boxed{M\frac{d\vec v}{dt} = \vec F_{ext} + \vec v_r\frac{dM}{dt}}$ Rocket Eq. Motion

$\vec v_r = \vec u - \vec v$: rel vel of Δm wrt Rocket

$M\frac{d\vec v}{dt} = M a_R$ Accel of Rocket.

F_{ext} : Ext Force, gravity, air resist

$\vec v_r\frac{dM}{dt}$ = Propulsive Force / Rocket Thrust.

Rate of transfer of $\vec P$ out of rocket system.

$v_r<0$ Rel. to rocket.

$\frac{dM}{dt}<0$ Mass decreases.

$d\vec v = \frac{F_{ext}}{M}dt + v_r\frac{dM}{M}$

Assume $F_{ext} = -Mg$ (g = const)

⑧

Lecture 18, Blackboard #5

$$\vec{\Delta P} = \vec{P}_f - \vec{P}_i = \cancel{M\vec{v}} + M\Delta\vec{v} - \Delta m\vec{v} - \cancel{\Delta m\,\Delta\vec{v}} + \vec{u}\,\Delta m - \cancel{M\vec{v}}$$

$$\vec{\Delta P} = M\,\vec{\Delta v} + (\vec{u}-\vec{v})\,\Delta m$$

$\Delta m = -\Delta M$ Let $\Delta t \to 0$

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} - (\vec{u}-\vec{v})\frac{dM}{dt}$$

$\boxed{M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{v}_r\frac{dM}{dt}}$ Rocket Eq. Motion

$\vec{v}_r = \vec{u} - \vec{v}$: rel vel of Δm wrt Rocket

$M\frac{d\vec{v}}{dt} = M a_R$ Accel of Rocket.

F_{ext} : Ext Force, gravity, air resist

$\vec{v}_r\frac{dM}{dt}$ = Propulsive Force / Rocket Thrust.

Rate of transfer of $\vec{P}$ out of rocket system.

$v_r < 0$ Rel to rocket

$\frac{dM}{dt} < 0$ Mass decreases.

$$d\vec{v} = \frac{F_{ext}}{M}dt + v_r\frac{dM}{M}$$

Assume $F_{ext} = -Mg$ (g = const)

(8)

Integrate: $\int_{v_0}^{v} dv' = -\int_0^t g\,dt' + v_r\int_{M_0}^{M}\frac{dM'}{M'}$

$$v = v_0 - gt + v_r \ln\frac{M(t)}{M_0}$$

v_0 = initial velocity

$M(t)$ = mass at time t.

v_r = exhaust velocity rel. to rocket

$(M_0 - M)$ = amount of fuel used.

Usually $gt \equiv$ small

Assume $v_0 = 0$

$$v = v_r \ln\frac{M(t)}{M_0}$$

$M(t)/M_0$	$\ln(M(t)/M_0)$
1/2	−0.69
1/10	−2.30
1/20	−3.00
1/100	−4.61

(9)