Rolling Motion of a Rigid Body

An important special type of rotational motion of a rigid body is one where the axis of rotation moves parallel to itself. Orientation of the axis in space does not change.

Translation + Rotation

- ball, cylinder, wheels on flat surfaces

Motion in which torques act to change the orientation of the rotation axis are much more complicated and we will leave out.

$$\left.\begin{aligned} \vec{P} &= M\frac{d\vec{R}}{dt} = \int \frac{d\vec{r}}{dt}\, dm \\ \frac{d\vec{P}}{dt} &= M\frac{dV}{dt} = M\frac{d^2R}{dt^2} = \sum \vec{F}_{ext} \\ K &= \frac{1}{2}MV^2 + \frac{1}{2}\int v_c^2\, dm \end{aligned}\right\} \text{Always true.}$$

For angular momentum we had that

$$\vec{L} = \vec{R}\times\vec{P} + \int_m (\vec{r}_c \times \vec{v}_c)\, dm$$

$$= \vec{L}_{or} + \vec{L}_s$$

$\vec{L}_{or}$ = Orbital angular momentum of cm about origin.

$\vec{L}_s$ = spin ang. mom. of object about axis through cm.

We also showed that

$$\frac{d\vec{L}_s}{dt} = \vec{\tau}_{cm}$$

where $\vec{\tau}_{cm}$ is the torque about the cm produced by the external forces.

– This latter result is independent of whatever the cm motion is – including acceleration. In this case the cm frame would be non-inertial.
[will not prove for 8.01]

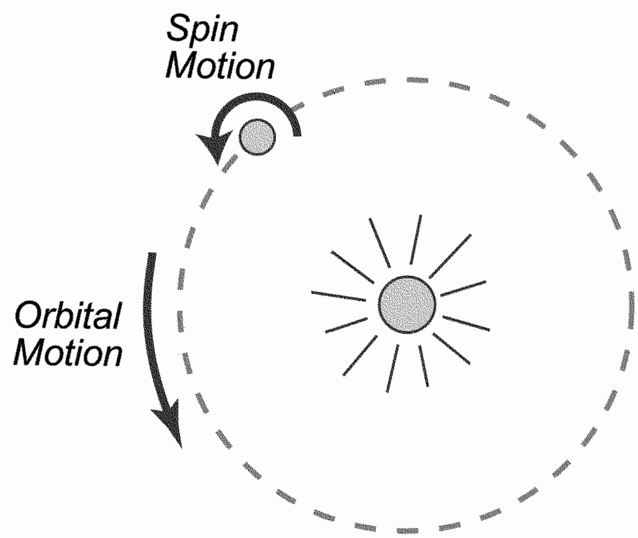
Rolling + Translation is a special case — orientation of $\vec{L}_s$ is constant in space. Magnitude may change due to applied torques. However, applied torque must be parallel to $\vec{L}_s$.

The object undergoes a general translation of the cm with a rotation about the cm constrained to an axis that moves only parallel to itself.

$$\tau_s = I_s \frac{d\omega}{dt} = I_s \alpha$$

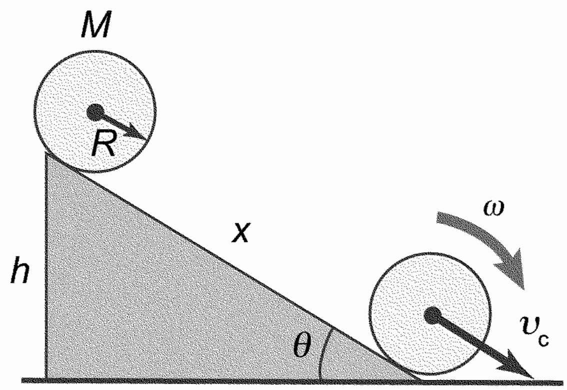
$$K = \frac{1}{2} M V^2 + \frac{1}{2} I_s \omega^2$$

Earth - Sun



Example: Rolling down Incline

- Release from rest at top.
- No slipping.
- Rolling possible only if friction present to produce torque about cm.
- No energy lost since contact point does not move relative to surface.



A round object rolling down an incline. Mechanical energy is conserved if no slipping occurs.

$$v_c = R\omega$$

$$K = \frac{1}{2} I_c \left(\frac{v_c}{R}\right)^2 + \frac{1}{2} M v_c^2$$

$$= \frac{1}{2}\left[\frac{I_c}{R^2} + M\right] v_c^2$$

Potential energy lost if object drops a height h:

$$\Delta U = Mgh.$$

$$\Delta K = \Delta U$$

$$\frac{1}{2}\left(\frac{I_c}{R^2} + M\right) v_c^2 = Mgh$$

$$v_c = \sqrt{\frac{2gh}{1 + I_c/MR^2}}$$

Example: Sphere down Plane

$$I_c = \frac{2}{5} MR^2$$

$$v_c = \sqrt{\frac{2gh}{1 + \frac{2}{5}\frac{MR^2}{MR^2}}} = \sqrt{\frac{10}{7} gh}$$

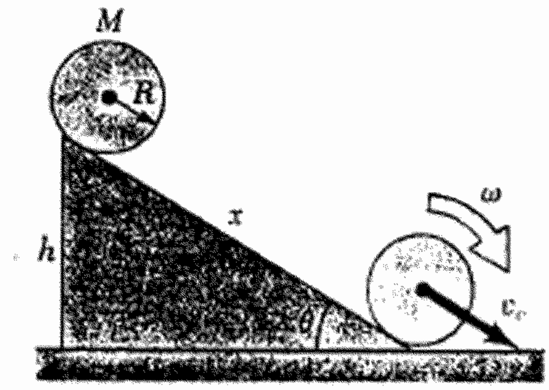


Figure A round object rolling down an incline. Mechanical energy is conserved if no slipping occurs.

x = distance along incline

$h = x \sin\theta$

$$v_c^2 = \frac{10}{7} g x \sin\theta$$

$$v_c^2 = 2 a_c x \quad \text{[constant acceleration]}$$

$$a_c = \frac{5}{7} g \sin\theta$$

Note:

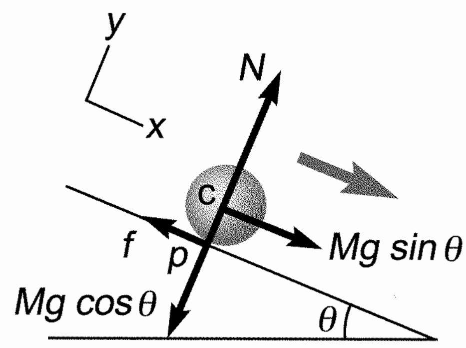
Velocity and acceleration are independent of mass and radius of sphere. All homogeneous solid spheres would have the same velocity and acceleration on a given incline.

Hollow spheres, cylinders + hoops would give similar results. Constants in expressions for v_c and a_c would be different.

Acceleration is less than for an object which does not roll.

Example: Rolling without slipping

Any round object of radius R rolls about its CM as it translates down plane of angle θ.



Free-body diagram for a solid sphere rolling down an incline.

Mass = M

Inertia = $I = \beta M R^2$

① $\sum \tau = I\alpha$ (aboot cm)

② $\tau_f + \tau_{mg} + \tau_N = Rf + 0 + 0 = I\alpha$

③ $\sum f_x = Mg\sin\theta - f = Ma_{cm}$

If motion is rolling without slipping

$$v_{cm} = R\omega \quad \text{and} \quad a_{cm} = R\alpha$$

$$Mg\sin\theta - \frac{I}{R}\alpha = Mg\sin\theta - \beta\frac{MR^2}{R}\frac{a_{cm}}{R}$$

$$= Mg\sin\theta - \beta M a_{cm} = Ma_{cm}$$

$$a_{cm} = \frac{g\sin\theta}{1+\beta}$$

Friction is static friction

$\therefore f_s \leq \mu_s N$

$$f_s = \frac{I\alpha}{R} = \frac{\beta m R^2}{R} \cdot \frac{1}{R} \frac{g \sin\theta}{1+\beta} \leq \mu_s M g \cos\theta$$

$$\therefore \tan\theta \leq \mu_s \frac{1+\beta}{\beta}$$

condition for angle above which object will slide as it rolls down the plane.

If object slides: $\left.\begin{array}{l} \omega R \neq v \\ \alpha R \neq a \end{array}\right\}$!!!

Hoop	$\beta = 1$
Cylinder	$\beta = 1/2$
Sphere	$\beta = 2/5$

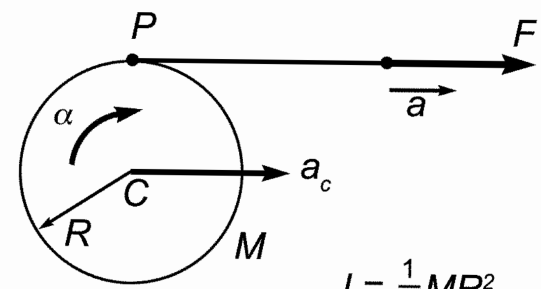
$$(a_{cm})_{sphere} = \frac{5}{7} g \sin\theta$$

$$(a_{cm})_{cyl} = \frac{2}{3} g \sin\theta$$

$$(a_{cm})_{Hoop} = \frac{1}{2} g \sin\theta$$

Example

A flat disk is on a flat frictionless surface. A force F is applied to the end of a string wrapped around the disk.
Disk rotates about vertical axis and translates horizontally.



Top View

$M = 2\,kg$
$R = 10\,cm$
$F = 5N$

a) Acceleration of cm.

$$a_c = \frac{F}{M} = \frac{5}{2} = 2.5\ m/s^2$$

b) Torques.

$$\alpha = \frac{\tau_c}{I_c} = \frac{FR}{\frac{1}{2}mR^2} = \frac{2F}{MR} = \frac{2\times 5}{2\times 0.10} = 50\ rad/s^2$$

c) what is $\vec{a}$ of free end of string?

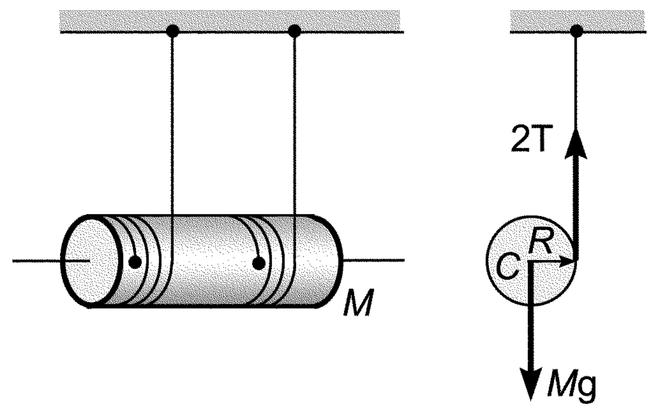
v_0 of string at point P is velocity of P rel. to cm ($v_T = R\omega$) plus the velocity of cm rel. to surface.

$$v_0 = R\omega + V$$

$$a_s = \frac{dv_0}{dt} = R\frac{d\omega}{dt} + a_c$$

$$= R\alpha + a_c = 7.5\ m/s^2$$

Example: Falling Cylinder



A string is wrapped around each end of a solid cylinder.
Cylinder is released and falls.

$$\tau_c = I_c \alpha$$

$$2TR = \frac{1}{2} M R^2 \alpha \qquad \text{[Rotation]}$$

$$Mg - 2T = Ma \qquad \text{[cm-motion]}$$

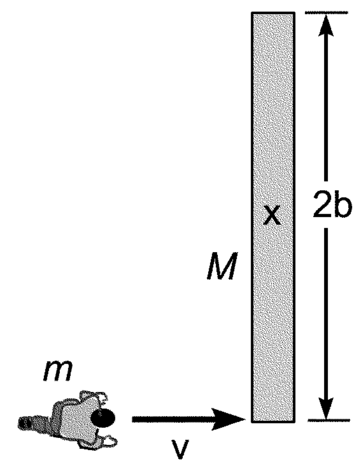
$$a = R\alpha \qquad \text{[String does not slip]}$$

Solve
$$a = \frac{2}{3} g$$

$$T = \frac{1}{6} Mg.$$

Example: Student and Plank

student mass $= m = 70\,kg$
Plank mass $= M = 50\,kg$
Long narrow plank $2b = 5m$.
Frictionless horizontal surface.



Student velocity $v = 3m/s$

Student jumps on to end of plank. What is the position of plank 1.2 s later?

Initial System: Plank + Running student.
Final System: (Plank + student) Rigid Motion

No Horizontal Forces: P_{Horiz} is Conserved

$$P_i = P_f$$
$$mv = (m+M)\,V$$

V: velocity of CM ~~of~~ the system.

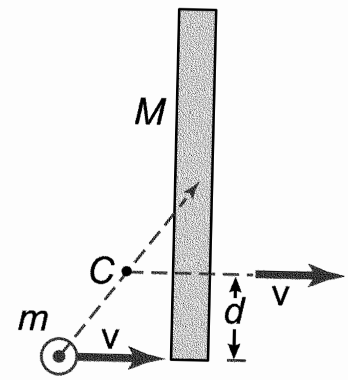
$$V = \frac{mv}{m+M} = \frac{70 \times 3}{70+50} = 1.75\ m/s$$

CM is located at a $\perp$ distance d from st.-line path of running student:

$$(m+M)\,d = M\,b$$

26-10

$$d = \frac{mb}{m+M} = \frac{50 \times 2.5}{70 + 50} = 1.04\,m$$



Since there are no external torques acting on the system the total angular momentum is conserved.
The angular momentum about a vertical axis through cm is:

$$L_i = L_f$$

$$L_i = mvd$$

$$L_f = I\omega = \Big[\underbrace{md^2}_{\text{Student - cm}} + \underbrace{M\left(\tfrac{1}{3}b^2 + (b-d)^2\right)}_{\text{Plank - cm}}\Big]\omega$$

cm, Parallel-Axis

$$\omega = \frac{mvd}{I} = \frac{mvd}{[\quad\quad]}$$

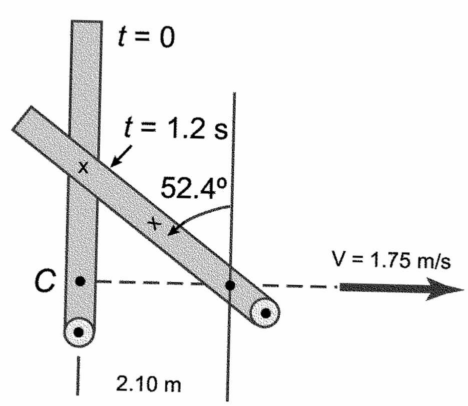
$$= \frac{70 \times 3 \times 1.04}{70 \times 1.04^2 + 50\left(\frac{1}{3}(2.5)^2 + (2.5 - 1.04)^2\right)}$$

$$= 0.762\ \text{rad/s} \qquad [46.3^\circ/\text{s}]$$

$\Delta t = 1.2\,s$

cm moves a distance (in st. line)

$vt = 1.75 \times 1.2 = 2.10\,m$



Plank + Student rotate through angle

$\theta = \omega t = 0.762 \times 1.2 = 0.914\,\text{rad}$

$= 52.4^\circ$ (ccw)

Rotational analogs of linear mechanical quantities and expressions

quantity	*linear*	*rotational*
displacement	$d\mathbf{r}$	$d\boldsymbol{\theta}$
velocity	$\mathbf{v} = \frac{d\mathbf{r}}{dt}$	$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}$
acceleration	$\mathbf{a} = \frac{d\mathbf{v}}{dt}$	$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$
constant acceleration	$v = v_0 + at$, etc.	$\omega = \omega_0 + \alpha t$, etc.
inertia	m	$I = \Sigma\, mr^2$
momentum	$\mathbf{p} = m\mathbf{v}$	$\mathbf{L} = I\boldsymbol{\omega} = \sum_{i=1}^{n} \mathbf{r} \times \mathbf{p}$
impulse	$\mathbf{P} = \mathbf{F}t$	$\mathcal{P} = \boldsymbol{\tau} t$
Newton's second law	$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$	$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = I\boldsymbol{\alpha}$
element of work	$dW = \mathbf{F} \cdot d\mathbf{r}$	$dW = \boldsymbol{\tau} \cdot d\boldsymbol{\theta}$
power	$P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$	$P = \frac{dW}{dt} = \boldsymbol{\tau} \cdot \boldsymbol{\omega}$
kinetic energy	$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$	$K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$

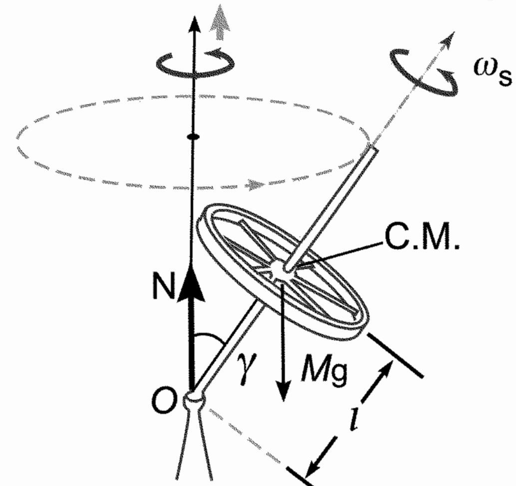
Pure rotational interrelations; R = perpendicular distance to axis of rotation:

$s = R\theta \qquad v = R\omega \qquad a = R\alpha \qquad \omega = 2\pi\nu$

Gyroscope Precession

Consider a spinning top supported at the origin O. Suppose it moves so that CM precesses about the vertical axis.

$\therefore\ N = Mg$ (no vert. motion)

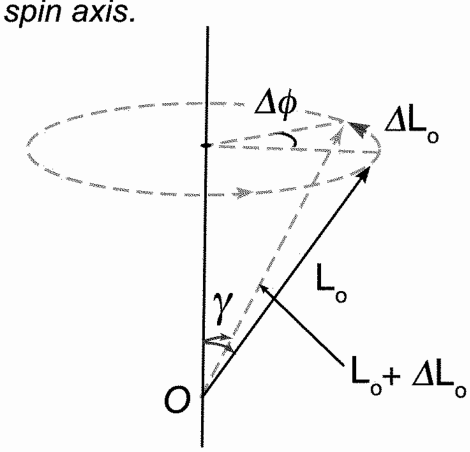


(a) The motion of a simple gyroscope about the frictionless bearing at *O*. The vertical axis is the *precessional axis*, and the axis of the top is the *spin axis.*

Torque about the origin O is

$$\vec{\tau}_o = \vec{R} \times \vec{F}$$
$$= mgl \sin\gamma$$

- magnitude
- direction always $\perp$ to $\vec{\omega}_s$ and $M\vec{g}$
- normal to plane defined by $\vec{\omega}_s$ and $\vec{\omega}_p = \vec{\Omega}$



(b) The change of the angular momentum during a time interval Δt.

Assume precessional motion is such that $\omega_p \ll \omega_s$, so that ang. momentum due to precession can be ignored.

$\therefore\ L_o \sim I\omega_s$ (Ang. mom. of spinning gyro).

$$\frac{d\vec{L}}{dt} = \vec{\tau}_o$$

In a time dt the change in ang. momentum is given by

$$d\vec{L} = \vec{\tau}\,dt = Mgl\sin\gamma\,(dt) \quad \text{[Ang. Impulse]}$$

The angle ($d\theta$) through which the axis swings in the time (dt) is

$$d\theta = \frac{dL}{L_o \sin\gamma} = \frac{Mg\ell \sin\gamma}{L_o \sin\gamma}(dt)$$

$$\Omega = \omega_p = \frac{d\theta}{dt} = \frac{Mg\ell}{L_o} \qquad \text{(precession frequency)}$$

Precession is independent of the angle of inclination and gyro can in fact be horizontal!

Note:

$$\omega_p = \frac{Mg\ell}{L_o}\frac{\sin\gamma}{\sin\gamma}$$

$$\omega_p L_o \sin\gamma = Mg\ell \sin\gamma$$

$$\boxed{\vec{\omega}_p \times \vec{L}_o = \vec{\tau}_o}$$

If the top is initially released with $\omega_p = 0$, it starts to fall. This gives rise to a torque giving a rotational displacement and the cm rises to its initial height.

In general cm undergoes cusplike motion

$\Longrightarrow$ Nutation

Too complicated for 8.01.

Gyroscope / Linear Dynamics

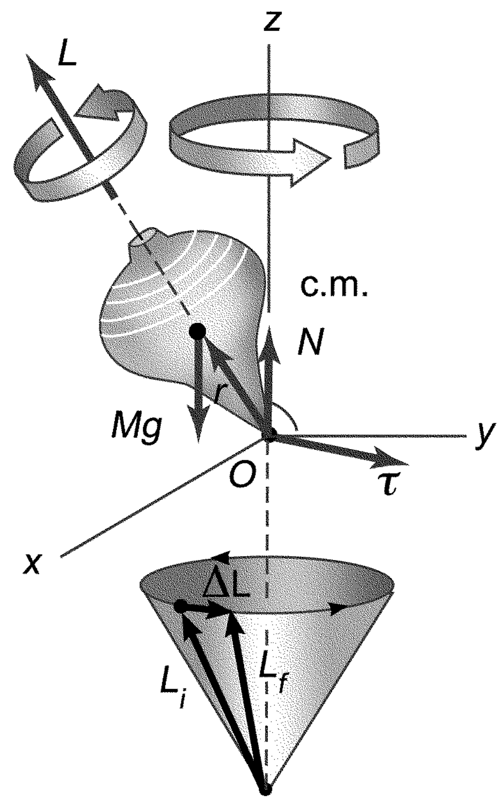
$\vec{L}_0 = I\vec{\omega}$ gyro spin ang. mom.

$\vec{\Omega} = \vec{\omega}_p$ angular precession

$\vec{\tau}$ torque.

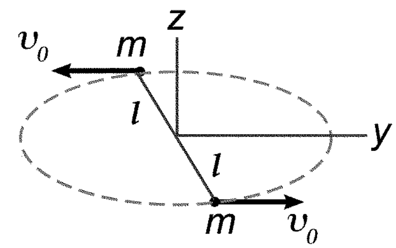
$$\vec{\tau} = \vec{\Omega} \times I\vec{\omega}$$

Gyro precession can be explained in terms of linear dynamics only.

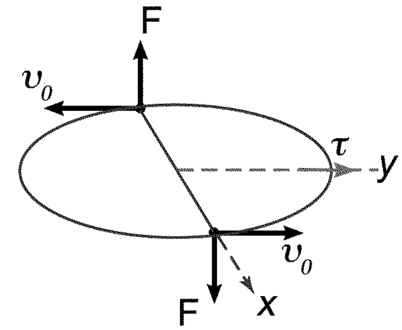


Precessional motion of a top spinning about its axis of symmetry. The only external forces acting on the top are the normal force, *N*, and the force of gravity, *Mg*. The direction of the angular momentum, *L*, is along the axis of symmetry.

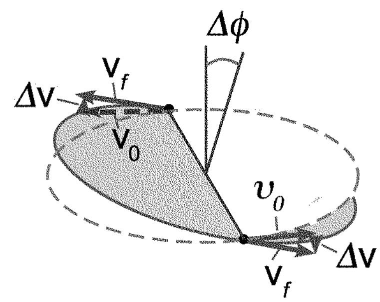
Gyroscope Precession



- Consider two particles of mass m
- Rigid massless rod of length $= 2L$
- Angular momentum L_s about z-axis
- Mass speed : v_0



Suppose torque applied during short time Δt while rod is along x-axis



$\Sigma (F + (-F)) = 0$

CM stays constant

Change in momentum of each mass:

$\vec{\Delta p} = m \vec{\Delta v} = \vec{F} \Delta t$ Impulse!

$\vec{\Delta v} \perp \vec{v_0}$

velocity changes direction and rod rotates about new direction.

Axis of rotation tilts slightly

$$\Delta \varphi \sim \frac{\Delta v}{v_0} = \frac{F \Delta t}{m v_0}$$ angle of tilt

Torque: $\tau = 2FL$

$L_s = 2m v_0 L$

length of rod

$$\therefore \Delta\varphi = \frac{F\Delta t}{m v_0} = \frac{2LF\Delta t}{2Lmv_0} = \frac{\tau \Delta t}{L_s}$$

$$\Omega = \frac{\Delta\varphi}{\Delta t} = \frac{\tau}{L_s}$$ Precession Frequency