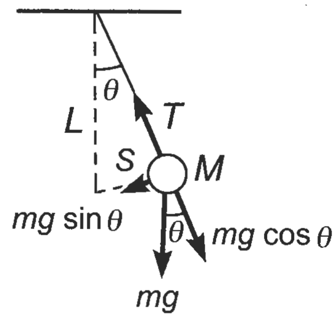
Simple Pendulum



When θ is small, the simple pendulum oscillates with simple harmonic motion about the equilibrium position (θ=0). The restoring force is *mg* sin θ, the component of weight tangent to the circle

Point mass m suspended on a massless cord of length L. Motion occurs in a vertical plane driven by the force of gravity.

Forces acting are the tension, $\vec{T}$, and the weight $m\vec{g}$. The tangential component of $m\vec{g}$ is a restoring force always acting toward $\theta = 0$.

$$F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

$$s = L\theta \qquad \text{(arc length)}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

For small θ: $\quad \sin\theta = \theta - \frac{\theta^3}{3!} + \cdots\cdots$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \qquad \theta \ll 1.$$

Solving: $\quad \theta = \theta_0 \cos(\omega t + \delta)$

$$\omega = \sqrt{g/L} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

The period and frequency of a simple pendulum depend only on the length of the string and the acceleration of gravity.

More exact solutions give T as a power series in terms of $\sin(\theta_0/2)$:

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \cdots\right]$$

$$\theta = 15^\circ$$

$$\frac{\theta - \sin\theta}{\theta} = 1.15\%.$$

Alternate : Torques

Torques relative to origin at top of string.

$$\tau = -mgL\sin\theta \qquad \text{(Restoring)}$$

Moment of inertia of pendulum

$$I = mL^2$$

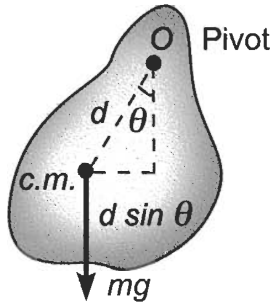
$$I\alpha = \tau \qquad \text{Rigid-Body Dynamics}$$

$$mL^2\frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Physical Pendulum

Any rigid body suspended from a fixed axis not passing through the cm. It will oscillate if displaced from its equilibrium position.



The physical pendulum consists of a rigid body pivoted at the point *O*, and not through the center of mass. At equilibrium, the weight vector passes through *O*, corresponding to $\theta = 0$. The resulting torque about *O* when the system is displaced through an angle θ is $mgd \sin\theta$.

Body pivoting a distance d from the cm. Torque provided by the force of gravity.

$$\tau = I\alpha$$

$$-mgd \sin\theta = I \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0 \qquad (\theta \ll 1)$$

$$\theta = \theta_0 \cos(\omega t + \delta) \qquad (\theta \ll 1)$$

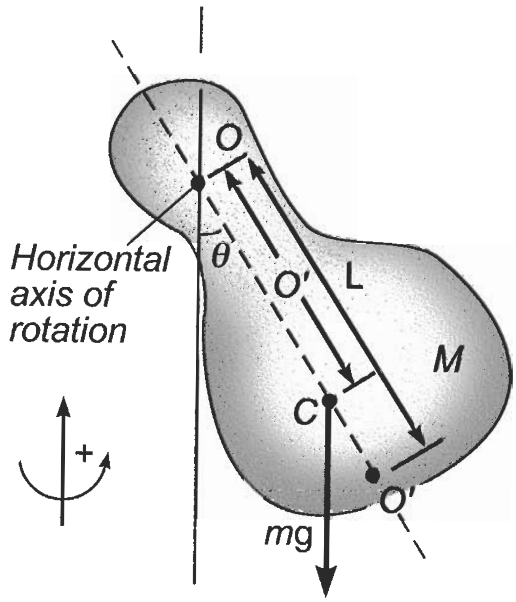
$$\omega = \sqrt{\frac{mgd}{I}} \quad \text{and} \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

An ideal pendulum of length

$$L = \frac{I}{md}$$

would have the same oscillation frequency.

Physical pendulum suspended from O oscillates as if it were an ideal pendulum with all its mass concentrated at O' a distance L from O on the extension through C.



Point O' is called the <u>center of oscillation</u> of the pendulum. O and O' are <u>conjogate points</u>.

Consider oscillations about O'.

$$T' = 2\pi \sqrt{\frac{I'}{mg(L-d)}}$$

$$L - d = \underbrace{\frac{I}{md}}_{=L} - d = \frac{I - md^2}{md} = \frac{I_c}{md} \qquad [I = I_c + md^2]$$

$I = I_c + md^2$

$I' = I_c + m(L-d)^2$

$$T' = 2\pi\sqrt{\frac{I'}{mg(L-d)}} = 2\pi\sqrt{\frac{I_c + m(L-d)^2}{mg(L-d)}}$$

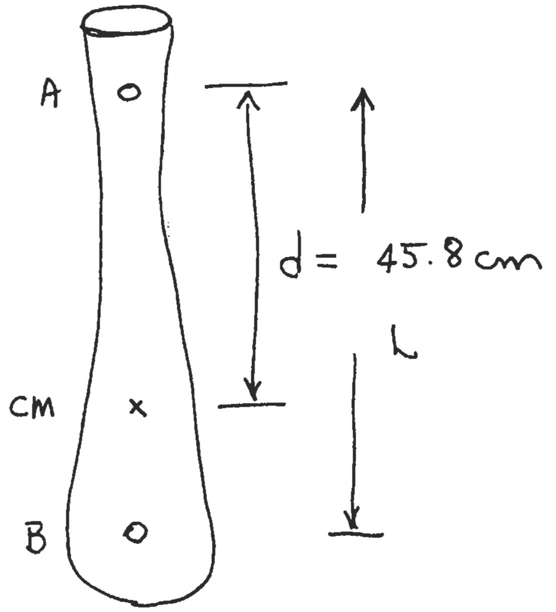
$$= 2\pi\sqrt{\frac{I_c + m\left(\frac{I_c}{md}\right)^2}{mg\left(\frac{I_c}{md}\right)}}$$

$$T' = 2\pi\sqrt{\frac{md^2 + I_c}{mgd}} = 2\pi\sqrt{\frac{I}{mgd}}$$

$$\therefore\ T' = T$$

[Periods are equal]

Example [Demo]



$M = 0.910$ kg.

$T_A = 1.5$ s

$$L = \frac{I}{md}$$

$$T_A = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{L}{g}}$$

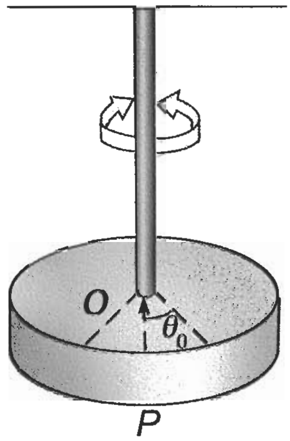
$$L = \left(\frac{T_A}{2\pi}\right)^2 g = \left(\frac{1.5}{2\pi}\right)^2 \times 9.81 = 55.9 \text{ cm.}$$

$T_B = T_A$ Conjugate Points

Torsional Pendulum

A rigid body is suspended by a wire attached at the top to a fixed support.

When the body is twisted through a small angle θ, the twisted wire exerts a restoring torque proportional to the angular displacement.



A torsional pendulum consists of a rigid body suspended by a wire attached to a rigid support. The body oscillates about the line *OP* with an amplitude θ_o.

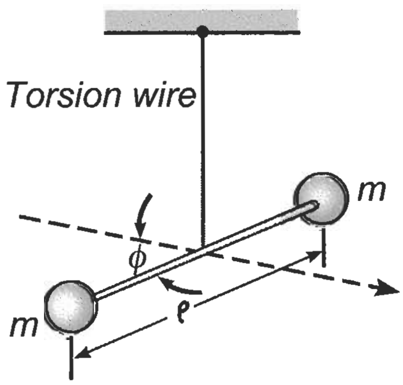
$$\tau = -K\theta$$

K: torsion constant of the wire.

$$\tau = -K\theta = I\frac{d^2\theta}{dt^2}$$

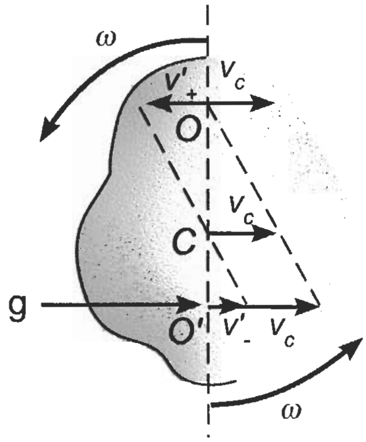
$$\frac{d^2\theta}{dt^2} + \frac{K}{I}\theta = 0$$

$$\omega = \sqrt{\frac{K}{I}} \quad \text{and} \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{K}}$$



- balance wheel of watch.
- galvanometers
- Cavendish torsional balance.

Percussion



Motion of a physical pendulum immediately after receiving an impulse g at O', the center of oscillation when suspended at O. At O, the speed $v = v_c - \omega v_c = O$; and at O', speed $v = v_c + \omega r'_c = (g/m)(1 + r'_c/r_c)$.

Body suspended from point O. Receives an impulse at O'.

$$\vec{g} = \Delta(m v_c) = \int \vec{F} dt$$

The resulting velocity of the cm is

$$v_c = \frac{\vec{g}}{m}$$

The impulse imparts an angular velocity to the body with respect to the cm of

$$\omega = \frac{g r'_c}{I_c} \qquad [\Delta L = \Delta(I_c \omega) = \tau dt = g r'_c]$$

The body <u>translates</u> and <u>rotates</u> about cm.

The linear velocity $\vec{v}$ at any point in the body is the vector sum of $\vec{v}_c$ and the tangential velocity $\vec{v}'$ about the cm due to its rotation.

At O' the velocities $\vec{v}'$ and $\vec{v}_c$ add. At O they oppose each other.

$$v_o = v_c - v' = v_c - \omega r_c = \frac{J}{m} - \omega r_c = \frac{J}{m}\left(1 - \frac{m r_c r_c'}{I_c}\right)$$

But $r_c r_c' = \dfrac{I_c}{m}$ so that $v_c = \omega r_c$

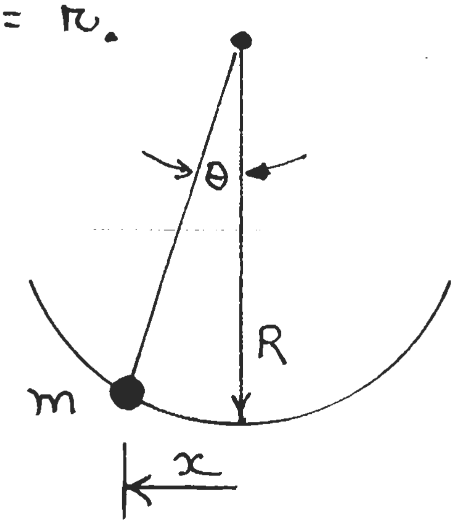
$$\therefore \; v_o = 0$$

object appears to be rotating about axis through O.

<u>Center of oscillation</u> is also called the <u>center of percussion</u>.

Example: Potential Well

- Particle of mass m rolling around the potential minimum of a cured surface of radius R. Sphere radius $= r$.



Assume: $\theta \ll 1$

$x \sim R\theta$

$$U(\theta) = mgR(1-\cos\theta)$$

$$\simeq mgR\theta^2/2$$

$$U(0) = 0$$

Let $\theta = \theta_0$ represent maximum amplitude of the displacement.

$x_0 = R\theta_0$ (maximum $-x$)

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + mgR\frac{\theta^2}{2} = mgR\frac{\theta_0^2}{2} \quad \text{[Cons of E]}$$

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + \frac{mg}{2R}x^2 = \frac{mg}{2R}x_0^2$$

Compare with:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2$$

Rewrite:

$$\frac{1}{2}\beta\frac{mr^2}{r^2}(r\omega)^2+\frac{1}{2}mv^2+\frac{mg}{2R}x^2=\frac{1}{2}\frac{mg}{R}x_0^2$$

$$\frac{1}{2}mv^2\left[\beta+1\right]+\frac{mg}{2R}x^2=\frac{1}{2}\frac{mg}{R}x_0^2$$

$$\omega^2=\frac{k}{m}=\frac{g}{R}\frac{1}{(\beta+1)}$$

$$T=2\pi\sqrt{\frac{R}{g}(\beta+1)}$$

[Demo]

$R = 90.6$ cm

$r =$

$T_{Thy} =$

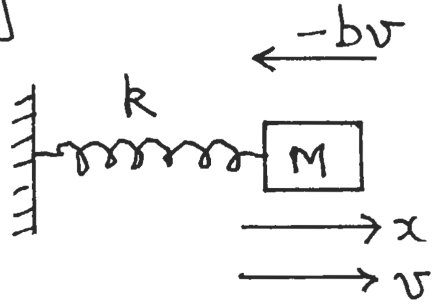
Damped Oscillations

- No friction
  - amplitude constant with time
- Real oscillators stop
  - pendula
  - mass/spring
- Dissipative / Frictional Forces
  ↳ damping
  - damped oscillation
  - assume damping force is proportional to velocity of body

$F = -bv$

↑ strength of damping

$$-kx - bv = m\frac{d^2x}{dt^2}$$



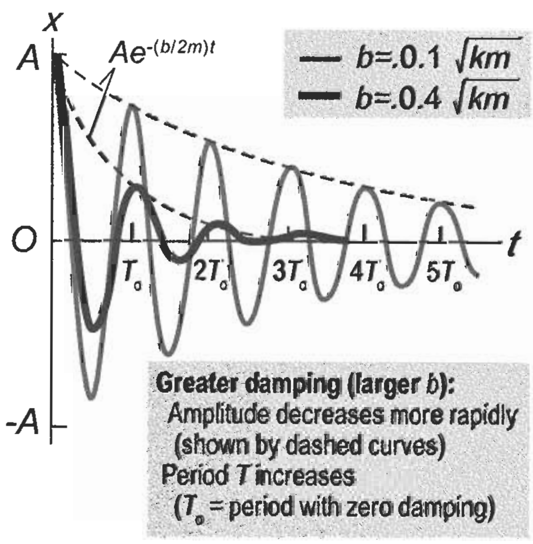
Solve de:

$$x = A_0 e^{-\frac{b}{2m}t} \cos(\omega' t + \varphi)$$

$$= A(t)\cos(\omega' t + \varphi)$$

$$\omega'^2 = \frac{k}{m} - \frac{b^2}{4m^2}$$

$$\omega'^2 < \omega_0^2 = \frac{k}{m}$$



- Amplitude decreases with time

$$E = \frac{1}{2} k A^2 \qquad \text{SHM, Total Energy.}$$

$$E(t) = \frac{1}{2} k A_0^2 \, e^{-\frac{b}{m}t}$$

$$E(t) = E_0 \, e^{-\frac{b}{m}t}$$

$\gamma = \dfrac{b}{m}$: measures

how long oscillator takes for energy to reach $1/e$ of its initial value.

<u>Critical Damping :</u>

Let $\quad \dfrac{k}{m} - \dfrac{b^2}{4m^2} = 0$

$$\Rightarrow b_c = 2\sqrt{km}$$

Recall : $\omega'^2 = \omega_0^2 - \dfrac{\gamma^2}{4}$

If $b > b_c$, $\omega'^2 < 0$

$\therefore$ No oscillation !!

$$x(t) = A_1 e^{-\left(\frac{\gamma}{2} + \beta\right)t} + A_2 e^{-\left(\frac{\gamma}{2} - \beta\right)t}$$

where, $\beta^2 = \dfrac{\gamma^2}{4} - \omega_0^2$

$$b = b_c = 2\sqrt{km}$$

$$x(t) = (A + Bt)\, e^{-\gamma t/2}$$

Critical damping important for mechanical systems. A constant force suddenly applied to a system which was at rest results in a new position of equilibrium with no overshoot or oscillation.

Note:

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$\frac{dE}{dt} = m v \frac{dv}{dt} + k x \frac{dx}{dt}$$

$$= v\,(ma + kx)$$

$$= -b v^2$$

If $b = 0$, no damping, $\frac{dE}{dt} = 0$; $E =$ constant

If $b \neq 0$, E decreases with time.