8.012 Physics I: Classical Mechanics
Fall 2008

Physics 8.012   Fall 2008

Exam 1

NAME: _____SOLUTIONS_____

Instructions:
1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL WORK**. Be sure to **circle your final answer**.
3. Read the questions carefully.
4. All work must be done in this booklet in workspace provided. Extra blank pages are provided at the end if needed.
5. NO books, notes, calculators or computers are permitted. **A sheet of useful equations is provided on the last page**.

Your Scores

Problem	Maximum	Score	Grader
1	10		
2	30		
3	20		
4	20		
5	20		
Total	100		

Problem 1: Quick Multiple Choice Questions [10 pts]

For each of the following questions circle the correct answer. You do not need to show any work.

(a) ~~Which of the following is **not** a valid force law?~~

$\vec{F} = b\dfrac{m_1 + m_2}{m_1 - m_2}\hat{r}_{12}$ where b has units of Newtons	$\vec{F} = 2Gmv^2\hat{x}$ where G is the gravitational force constant	$\vec{F} = b\ddot{r}\hat{\theta} + m\dfrac{v^2}{r}\hat{r}$ where b has units of kg	$\vec{F} = b\dfrac{mx}{t^2}\hat{z}$ where b has units of radians

Both of these solutions are right, the first because it doesn't satisfy Newton's 3rd Law (switch 1 and 2 and you don't get equal and opposite), the second because of units

(b) A tire rolls on a flat surface with constant angular velocity Ω and velocity $\vec{V}$ as shown in the diagram to the right. If <u>V > ΩR</u>, in which direction does friction from the road act on the tire?
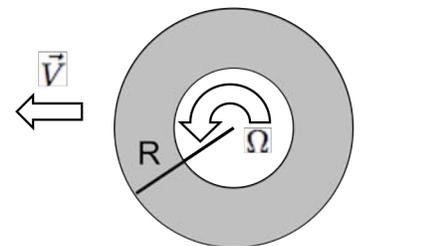
Toward the left	Toward the right	Friction does not act on the tire

The intention was to have friction spinning the wheel up, but because of the word "constant" in the question, we deemed this question to ambiguous so it wasn't counted in the final score

(c) You pull a brick of mass M sitting on a flat table using a thick rope of mass m. Friction between the surface of the table and the brick is sufficient so that the brick remains at rest. Compared to the force with which you pull on one end of the rope, the force that the other end of the rope exerts on the brick is:
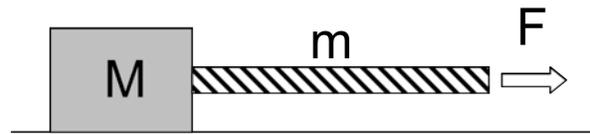
Less	Greater	The same	Zero

There is no net force on the rope (or block), so there cannot be a difference in tension along the rope. Hence the forces are the same at both ends.

(d) As a swinging pendulum passes through its lowest point, in which direction does the total net force act?

| Only in an angular direction	Only in a radial direction	In both angular and radial directions	There is zero net force at the equilibrium point

Because the mass is moving in circular motion, there must be a radial force. At the lowest point there are no net angular forces, however.

(e) A pendulum with mass M and length L is released as a small angle off of vertical and oscillates with period P. If we double the mass and halve the length of the pendulum then the new period is

$2P$	$\sqrt{2}P$	P	$P/\sqrt{2}$	$P/2$

Doubling the mass does not affect the period but decreasing the length decreases the period, which is $2\pi(l/g)^{0.5}$

Problem 2: The Accelerated Atwood Machine [30 pts]
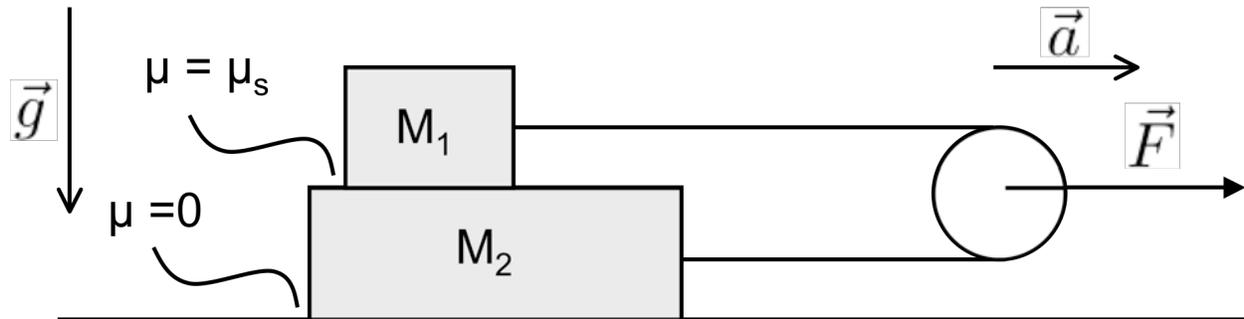
Two blocks of masses M_1 and M_2 ($M_2 > M_1$) are stacked on top of each other and start at rest on the surface of a <u>frictionless</u> table. The masses are connected via an ideal pulley (massless string and nearly massless pulley wheel), and the coefficient of static friction (assumed equal to the coefficient of kinetic friction) between the block surfaces is μ_S. The pulley is accelerated to the right by a force $\vec{F}$, resulting in an acceleration of the pulley wheel of $\vec{a}$. Assume that gravity acts with constant acceleration g downward.

(a) [5 pts] Draw force diagrams for each of the blocks and the pulley wheel, clearly indicating all horizontal and vertical forces acting on them.

(b) [5 pts] If the blocks do not slip relative to each other, what are their accelerations?

(c) [10 pts] Assume that the blocks do slip relative to each other. Determine each block's horizontal acceleration as a function of the parameters specified above (i.e., M_1, M_2, μ_S, g, a and F). Which block has a higher acceleration? Be sure to work in an inertial reference frame!

(d) [10 pts] What is the minimum force F required to cause one block to slip relative to the other? Assume that the mass of the pulley is negligible compared to those of the blocks.

(a) The force diagrams are as shown below – note that the weight of the pulley is specifically excluded here and the string tension assumed to be constant because the string is massless (no points were taken off for not assuming these things). Common errors were not matching up force pairs; i.e., F_f (crucial!) and N_1
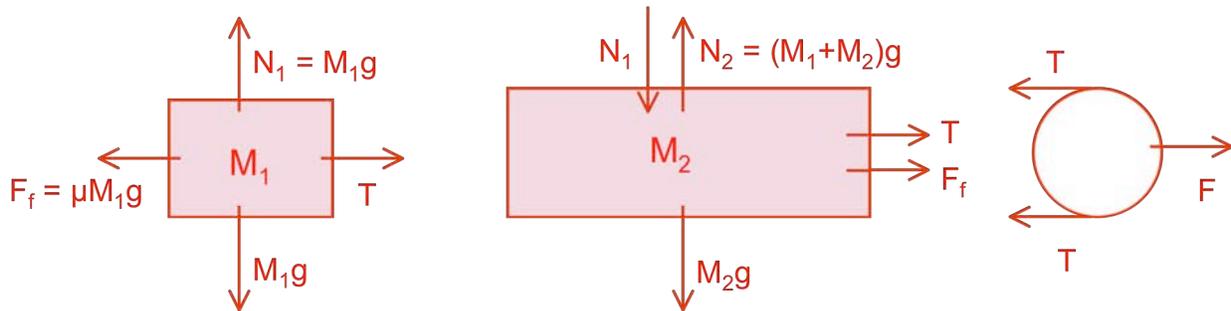
Note that the direction of the friction force takes some thought, but can be determined if one considers the problem without friction. In that case the smaller mass M_1 would move to the right faster (same tension force but smaller mass); hence, friction acts to stop that relative motion by acting toward the left on M_1. Newton's 3rd law then states that the same friction force acts toward the right on M_2.

(b) If the blocks do not slip, then their accelerations are exactly equal to that of the pulley, a. This can be shown formally through the constraint equation:

$$(x_p - x_1) + (x_p - x_2) = constant$$

$$\Rightarrow 2\ddot{x}_p = 2a = \ddot{x}_1 + \ddot{x}_2$$

and the condition

$$\boxed{\ddot{x}_1 = \ddot{x}_2 = a}$$

Because F is also specified, one can also determine solve for the equations of motion assuming that the two accelerations are the same and equally determine:

$$\boxed{\ddot{x}_1 = \ddot{x}_2 = \frac{F}{M_1 + M_2}}$$

$$\ddot{x}_1 = \ddot{x}_2 = 2\mu g \frac{M_1}{M_1 + M_2}$$

although care must be taken here since $\mu \leq \mu_S$.

(c) Based on the force diagram above, the equations of motion are:

$$M_1 \ddot{x}_1 = T - \mu_S M_1 g$$

$$M_2 \ddot{x}_2 = T + \mu_S M_1 g$$

Because the pulley is effectively massless, $M_p a_p \approx 0$ and hence $F = 2T$. This alone provides expressions for the two pulleys:

$$\ddot{x}_1 = \frac{F}{2M_1} - \mu_S g$$

$$\ddot{x}_2 = \frac{F}{2M_2} + \mu_S \frac{M_1}{M_2} g$$

We can also solve for the rope tension using the constrain equation:

$$\ddot{x}_1 + \ddot{x}_2 = 2a = T(\frac{1}{M_1} + \frac{1}{M_2}) + \mu_S g(\frac{M_1}{M_2} - 1)$$

$$\Rightarrow T = 2a \frac{M_1 M_2}{M_1 + M_2} + \mu_S M_1 g \frac{M_2 - M_1}{M_1 + M_2}$$

substituting back in one can derive

$$\ddot{x}_1 = 2a \frac{M_2}{M_1 + M_2} - 2\mu_S g \frac{M_1}{M_1 + M_2}$$

$$\ddot{x}_2 = 2a \frac{M_1}{M_1 + M_2} + 2\mu_S g \frac{M_1}{M_1 + M_2}$$

note that $\ddot{x}_1 + \ddot{x}_2 = 2a$.

A third pair of valid expressions (in terms of F and a) are

$$\ddot{x}_1 = \frac{F - 2M_1 a}{M_2 - M_1}$$

$$\ddot{x}_2 = \frac{2M_2 a - F}{M_2 - M_1}$$

again note that $\ddot{x}_1 + \ddot{x}_2 = 2a$.

(d) There are number of ways to consider this problem, but the most obvious is to consider the point at which $\ddot{x}_1 > a$ (prior the slipping both masses are accelerating with a). This yields any of the following conditions based on the answers to part (c):

$$F > 2M_1 g(\mu_S + 1)$$

$$F > (M_1 + M_2)a$$

$$F > 2\mu_S M_1 g \frac{M_1 + M_2}{M_2 - M_1}$$

Problem 3: Hanging Rope [20 pts]
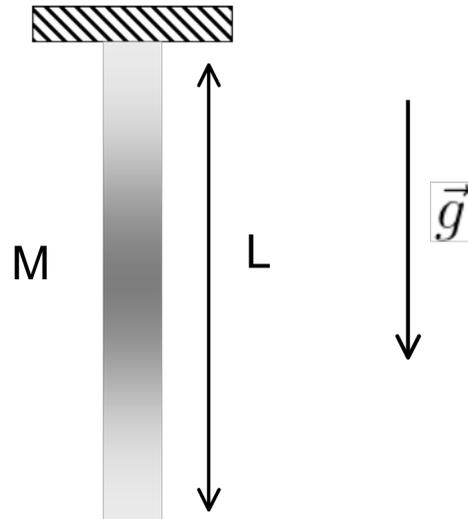
Consider a rope of total mass M and length L suspended at rest from a fixed mount. The rope has a linear mass density that varies with height as $\lambda(z) = \lambda_0\sin(\pi z/L)$ where λ_0 is a constant. Constant gravitational acceleration g acts downward.

(a) [5 pts] Determine the constant λ_0.

(b) [5 pts] What is the tension force at the free (bottom) end of the rope?

(c) [10 pts] Calculate the tension along the rope as a function of distance z below the mount.

(a) The constant can be found by noting that the integral of the linear density over the length of the rope should equal the mass; i.e.,

$$M = \int_0^L \lambda(z)dz = \lambda_0 \int_0^L \sin\frac{\pi z}{L}dz = \frac{2}{\pi}L\lambda_0$$

$$\Rightarrow \boxed{\lambda_0 = \frac{\pi}{2}\frac{M}{L}}$$

(b) Because there is nothing hanging from the bottom end, the tension force is simply $\boxed{0.}$

(c) There are two ways of determining this. First, you can consider that the tension at any height z is simply that required to hold up the mass below z; i.e.,

$$T(z) = g\int_z^L \lambda(z)dz = \frac{\pi}{2}\frac{Mg}{L}\int_z^L \sin\frac{\pi z}{L}dz = \boxed{\frac{Mg}{2}\left(1 + \cos\frac{\pi z}{L}\right)}$$

Alternately, one can set up a differential equation at any height z requiring that the difference in tensions must support the differential mass at that point (note that z increases downward); i.e.,

$$T(z + dz) - T(z) = dT = dMg = \lambda(z)gdz$$

$$\Rightarrow -\int_{T(z)}^{T(L)} dT = T(z) = g\int_z^L \lambda(z)dz$$

which reduces to the same result. Note that here we have used the result from (b) that T(L) = 0.

Problem 4: Don't Slip! [20 pts]
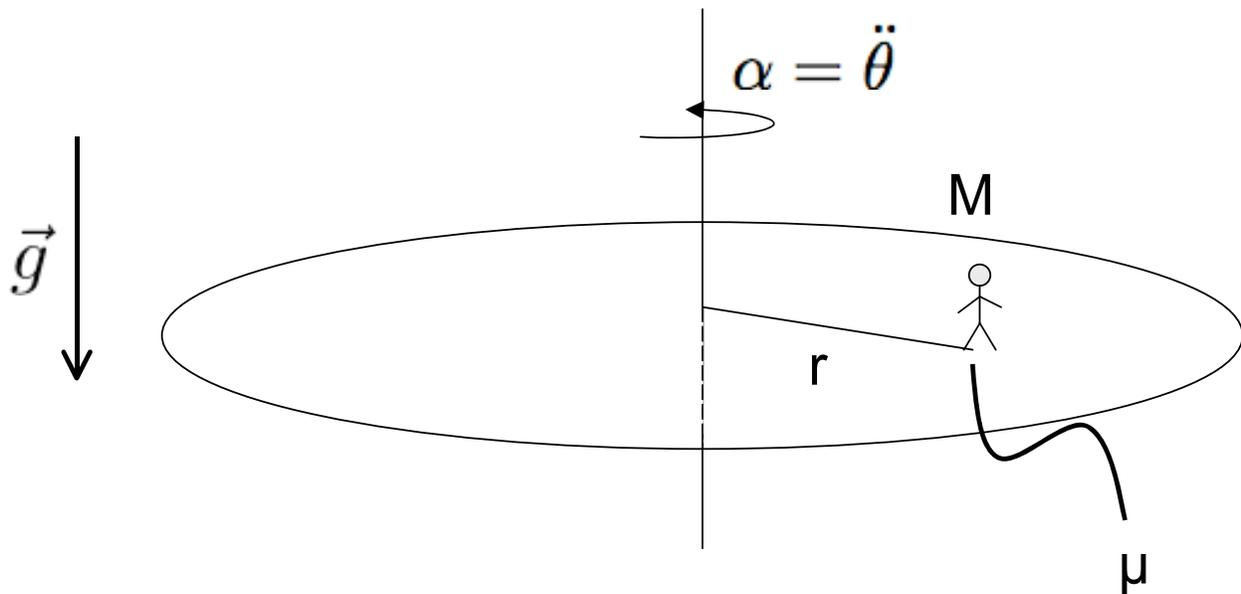
An 8.012 student of mass M stands on a rigid disk at a distance r from the center axis. Assume that the coefficient of friction between the student's shoes and the disk surface is μ. At time t = 0, the disk begins to rotate with a constant angular acceleration rate $\alpha = \ddot{\theta}$. Assume that gravity acts with constant acceleration g downward.

(a) [5 pts] What is the maximum value of angular acceleration rate (α_{max}) such that the student does not immediately slip?

(b) [10 pts] Assuming that $\alpha < \alpha_{max}$, what is the total friction force acting on the student as a function of time (prior to slipping)? Write your answer as a vector in polar coordinates.

(c) [5 pts] Assuming that $\alpha < \alpha_{max}$, how long after the disk starts rotating will the student slip?

(a) Recall that the force acting on the student in polar coordinates is expressed as:

$$\vec{F} = M(\ddot{r} - r\dot{\theta}^2)\hat{r} + M(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

at t=0, $\dot{\theta} = \dot{r} = \ddot{r} = 0$, so the force expression reduces to:

$$\vec{F} = Mra\hat{\theta}$$

we want the case where the student doesn't slip, so

$$|\vec{F}| < F_f(max) = \mu M g$$

$$\Rightarrow \boxed{\alpha < \frac{\mu g}{r}}$$

(b) Now we have to explicitly consider the angular rate as a function of time, although note that the radius does not change so $\dot{r} = \ddot{r} = 0$ while $\dot{\theta} = \alpha t$. Then

$$\boxed{\vec{F} = -Mra^2t^2\hat{r} + Mra\hat{\theta}}$$

(c) Again we want to satisfy the condition:

$$|\vec{F}| < F_f(max) = \mu M g$$

$$M\sqrt{r^2\alpha^4t^4 + r^2\alpha^2} < \mu M g$$

$$\Rightarrow r^2\alpha^4t^4 < \mu^2 g^2 - r^2\alpha^2$$

$$\Rightarrow \boxed{t < \frac{1}{\alpha}(\frac{\mu g^2}{r} - \alpha^2)^{1/4}}$$

Problem 5: Will the LHC Destroy the World? [20 pts]

There has been some speculation that the Large Hadron Collider (or LHC), a particle accelerator experiment that will smash protons together at incredible energies, might create a small black hole that could devour the Earth. Let's imagine that the LHC does create such a black hole of mass M (assume initially than M << M_{Eartg}). Once it has formed (initially at rest), the black hole will immediately fall under the influence of gravity toward the center of the Earth. Assume for this problem that the Earth has a constant volume mass density ρ, and that the black hole doesn't eat that much of the Earth as it passes through it.

(a) [5 pts] Using dimensional analysis, come up with an order-of-magnitude expression for the oscillation period of the black hole as it passes through the Earth based on the relevant parameters of this problem.

(b) [10 pts] Now show that the black hole undergoes simple harmonic motion (similar to spring motion) by determining its acceleration as a function of time, and derive an expression for the oscillation period. How does this compare with your answer in part (a)?

(c) [5 pts] The Earth does not have a constant mass density but rather gets denser closer to its center. Assuming that it is still spherically symmetric, would this fact cause the oscillation period of the black hole to be shorter or longer? Justify your answer.

(d) [10 pts **BONUS**] Assume that the particle collision that creates the black hole converts the equivalent of an entire year's worth of global energy production (about 5×10^{20} Newton-meters) into a single black hole at rest. Also, the radius of a black hole is the distance at which the gravitational escape velocity exceeds the speed of light. Based on these assumptions, use dimensional analysis to estimate the mass and radius of the black hole to within an order of magnitude. Based on these values, should we worry about an LHC black hole? Some important numbers for this question are the gravitational constant, $G \approx 7 \times 10^{-11}$ m^3 kg^{-1}s^{-2}; the speed of light, $c \approx 3 \times 10^8$ m/s; and, for comparison, the radius of a proton, $r_{proton} \approx 10^{-15}$ m, and the radius of the Earth, $R_{Earth} \approx 6 \times 10^6$ m.

(a) The important parameters are the mass of the Earth ($M_E \sim [M]$), the radius of the Earth ($R_E \sim [L]$), the density of the Earth ($\rho \sim [M/L^3]$) and the gravitational constant ($G \sim [L^3 T^{-2} M^{-1}]$). Note that the mass of the black hole shouldn't matter based on our knowledge of the gravitational force law (i.e., we are looking for an aceeleration, which is independent of black hole mass). We are seeking a period $\sim$ [T], and the combinations that work are:

$$P \sim \sqrt{\frac{R_E^3}{GM_E}}$$

$$P \sim \frac{1}{\sqrt{G\rho}}$$

However, ρ and M_e/R_e^3 are degenerate with each other, hence either expression is viable.

(b) We want to compute the force on the black hole as a function of radius from the center of the Earth using two important gravitational results: (1) the gravitational force outside a spherical symmetric mass is equivalent to the force from a point mass and (2) inside a spherically symmetric shell an object feels no net gravitational force. Hence the black hole is only pulled in by the fraction of the Earth's mass that is interior to its radial position:

$$\vec{F} = M\ddot{r}\hat{r} = -G\frac{MM_E(r)}{r^2}\hat{r}$$

where

$$M_E(r) = \int_0^r \rho(r)dV = 4\pi\rho \int_0^r r^2 dr = \frac{4\pi}{3}\rho r^3 = M_E\frac{r^3}{R_e^3}$$

$$\Rightarrow \ddot{r} = -G\frac{M_E}{R_E^3}r \equiv -\omega^2 r$$

which has the form of a simple harmonic oscillator with period

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{GM_e}} = \sqrt{\frac{3\pi}{G\rho}}$$

(c) The period will generally be <u>shorter</u> since the force (and hence acceleration) on the black hole will be greater at a given radius as there is more total mass within that radius. In the extreme case of a infinitesimal shell of mass the period can become very long as the BH will experience no acceleration past the thin shell (and that acceleration will be the same as the case of a constant density ball).

(d) For the black hole mass we can use Einstein's equation to relate energy to mass (an approximation to be sure), so m ≈ E/c^2 ≈ 5×10^{20} N-m/(9×10^{16} m^2/s^2) ≈ 5000 kg, or about 5 tons.

For the radius, the relevant quantities are the black hole mass ([M]), G ([L^{3}T^{-2}M^{-1}]) and c ([L^2/T^2]), which can be combined as

$$R \sim \frac{GM}{c^2}$$

to give a quantity that has dimension length. With the values given, this gives a radius of roughly 5×10^{-24} m – about 9 orders of magnitude smaller than a proton! So I won't worry…

$$\boxed{\text{USEFUL EQUATIONS}}$$

Trajectory for constant acceleration $\vec{a}$

$$\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}(0)t + \vec{r}(0)$$

Velocity in polar coordinates

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Acceleration in polar coordinates

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Taylor expansion of function f(x):

$$f(x) = f(a) + \frac{1}{1!}\frac{df}{dx}|_a(x-a) + \frac{1}{2!}\frac{d^2f}{dx^2}|_a(x-a)^2 + \dots$$

Newton's Gravitational Law

$$\vec{F}_{12} = -G\frac{m_1m_2}{r_{12}^2}\hat{r}_{12}$$