8.022 Lecture Notes Class 42 - 12/6/2006

1-D:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad f(x,t)$$

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad f(\vec{x},t)$$

Wave equation can be extended!

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}(z,t) \qquad \text{f a vector function}$$

transverse: $\hat{n}\cdot\hat{z} = 0 \longleftarrow$ EM waves
longitudal: $\hat{n}\times\hat{z} = 0 \longleftarrow$ sound waves

$$\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} \Rightarrow S \perp E, B}$$

Assume $\hat{n} \neq \hat{n}(z,t)$ for now.

$$\hat{n} = \cos\theta\hat{x} + \sin\theta\hat{y}$$

$$\tilde{f}(z,t) = \tilde{A}\cos\theta e^{i(kz-\omega t)}\hat{x} + \tilde{A}\sin\theta e^{i(kz-\omega t)}\hat{y}$$

θ is the polarization angle of the wave

No local charge and current:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E \\ &= \nabla \times (-\tfrac{\partial B}{\partial t}) \\ &= -\tfrac{\partial}{\partial t}(\nabla \times B) \\ &= -\mu_0 \epsilon_0 \tfrac{\partial^2 E}{\partial t^2} \end{aligned}$$

So,

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad || \quad \nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\Longrightarrow \frac{1}{v^2} = \epsilon_0 \mu_0$$

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

Since these E and B waves travel at the speed of light. it must be what light is made of.

Since

$$\vec{\nabla} \cdot \vec{E} = 0, (\tilde{E}_0)_z = 0$$

Assumptions

- ω - given frequency $\Leftrightarrow$ monochromatic
- plane wave in $\hat{z}$ - direction

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}$$
$$\tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

$$\vec{\nabla} \cdot \vec{B} = 0, (\tilde{B}_0)_z = 0$$

this means waves are transverse waves

What about $\vec{\nabla} \times E$?

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$-k(\tilde{E}_0)_y \hat{x} + k(\tilde{E}_0)_x \hat{y} = \omega(\tilde{B}_0)_x \hat{x} + \omega(\tilde{B}_0)_y \hat{y}$$
$$\Longrightarrow \tilde{B}_0 = \frac{k}{\omega}(\hat{z} \times \hat{E}_0)$$

Tells us 2 things !

- $\vec{B} \perp \vec{E}$
- In phase !
  (we can get from B_0 to E_0 merely by cross-product)

$$|B_0| = \tfrac{k}{\omega}|\vec{E}_0| = \frac{\frac{2\pi}{\lambda}}{2\pi v} = \tfrac{1}{v}|\vec{E}_0|$$
$$|B_0| = \tfrac{1}{c}|\vec{E}_0|$$

Extend to 3-D:

$$\begin{aligned} \tilde{E}(\vec{r},t) &= \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{n} \\ \tilde{B}(\vec{r},t) &= \tfrac{\tilde{E}_0}{c} e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\vec{v}\times\vec{n}) \end{aligned}$$

<u>Energy, Momentum</u>

$$\begin{aligned} U &=^{[SI]} \tfrac{1}{2}[\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2] =^{[cgs]} \tfrac{E_0^2+B_0^2}{8\pi} \\ &= \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2}\tfrac{1}{\mu_0}\mu_0\epsilon_0 E^2 \\ U &= \epsilon_0 E^2 \\ &= \epsilon_0 \cos^2(\vec{k}\cdot\vec{r}-\omega t+\delta)E_0^2 \end{aligned}$$

Energy in EM wave is equally distributed

- Poynting time!

$$\begin{aligned} \vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} &= \tfrac{1}{mu_0}\hat{z}\cdot E_0\cos(...)\cdot\tfrac{E_0}{c}\cos(...) \\ &= \hat{z}\cdot E_0^2\cos^2(...)\tfrac{1}{\mu_0}(\sqrt{\epsilon_0\mu_0})^2\cdot c \\ &= \hat{z}\cdot c\epsilon_0 E_0^2\cos^2(\vec{k}\cdot\vec{r}-\omega t+\delta) \end{aligned}$$

Know $\hat{z}\cdot c = \vec{v}$ and $U = \epsilon_0 E_0^2\cos^2(\vec{k}\cdot\vec{r}-\omega t+\delta)$

$$\vec{S} = u\cdot\vec{v}$$

- Momentum

$$\vec{p} = \frac{\vec{S}}{c^2} \qquad (\text{ from eq. } \vec{\mathbf{P_{em}}} = \mu_0\epsilon_0 \int_V \vec{S} d\tau)$$
$$= \qquad \frac{1}{c} \cdot u\hat{z} \qquad (\text{ Let } \hat{z} = \vec{v})$$

$$< u >= \quad \frac{1}{2}\epsilon_0 E_0^2$$
$$< \vec{S} >= \quad \frac{1}{2}c\epsilon_0 E_0^2 \hat{z}$$
$$< \vec{p} >= \quad \frac{1}{2}\frac{1}{c}\epsilon_0 E_0^2 \hat{z}$$
$$I = < |S| >= \frac{1}{2}c\epsilon_0 E_0^2$$

Only difference between energy , $\vec{S}$, and momentum: factors of c.