Class 26: Outline

Hour 1:

Driven Harmonic Motion (RLC)

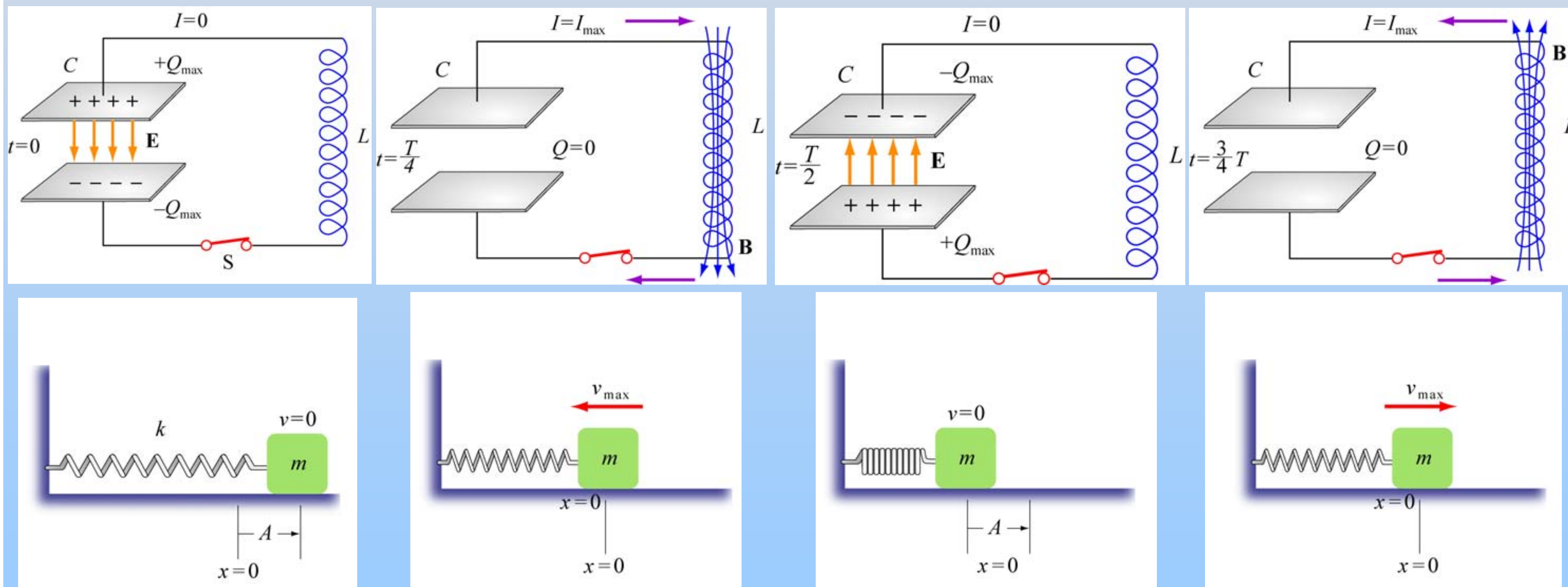
Hour 2:

Experiment 11: Driven RLC Circuit

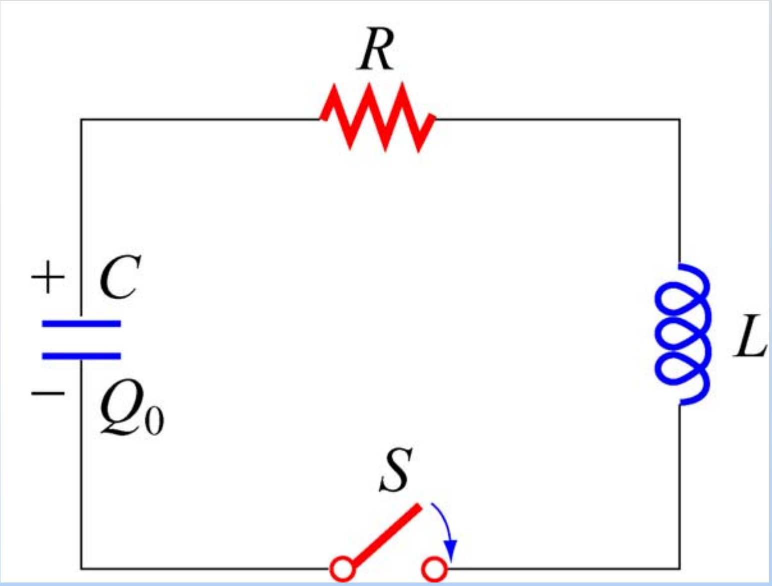
Last Time:
Undriven RLC Circuits

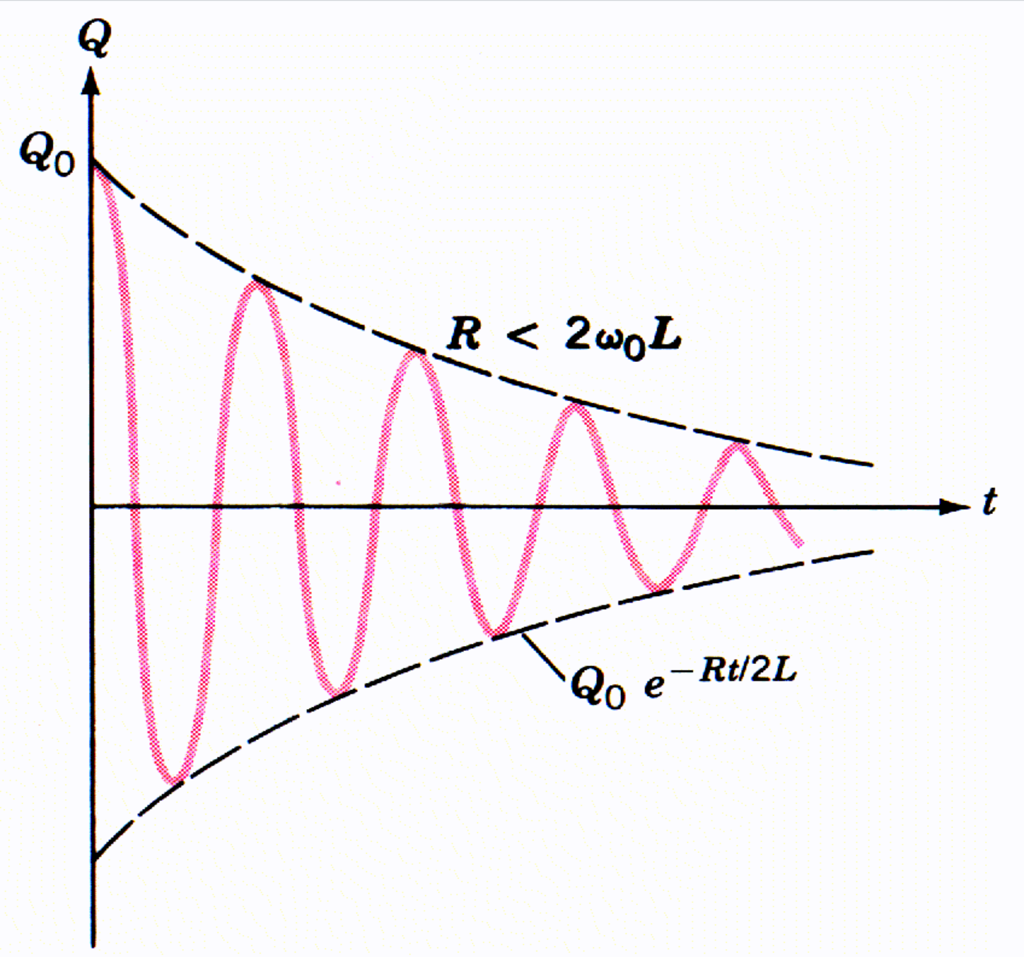
LC Circuit

It undergoes simple harmonic motion, just like a mass on a spring, with trade-off between charge on capacitor (Spring) and current in inductor (Mass)



Damped LC Oscillations

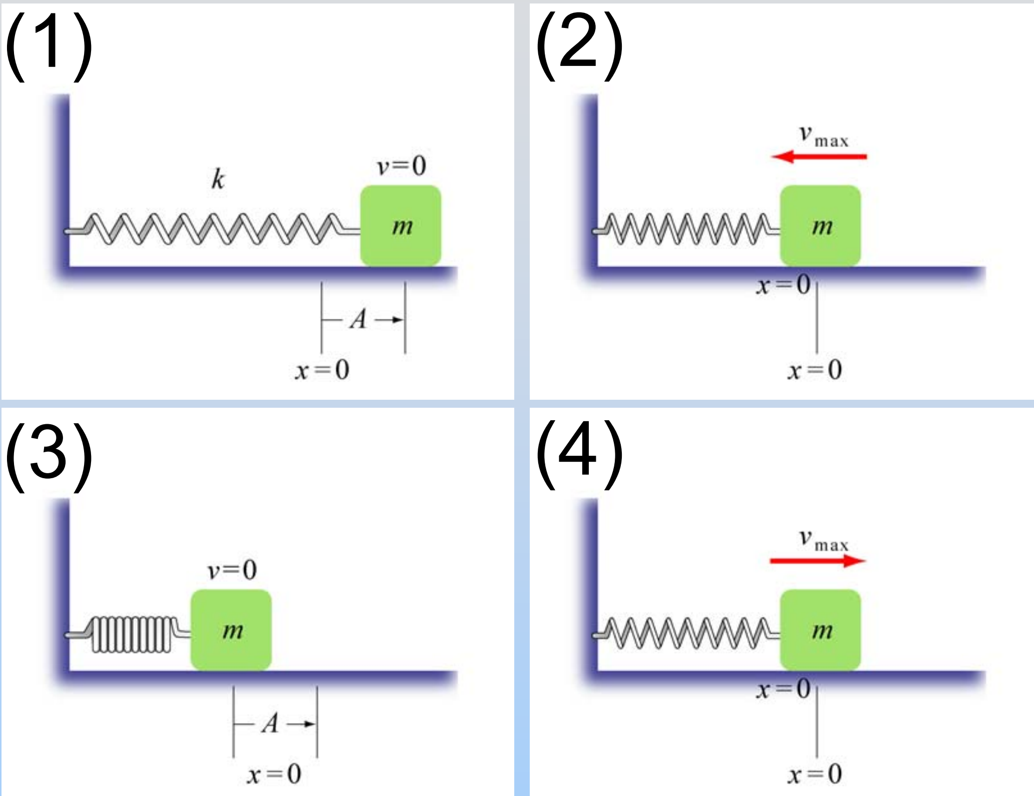




Resistor dissipates energy and system rings down over time

Mass on a Spring: Simple Harmonic Motion` A Second Look

Mass on a Spring



We solved this:

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Simple Harmonic Motion

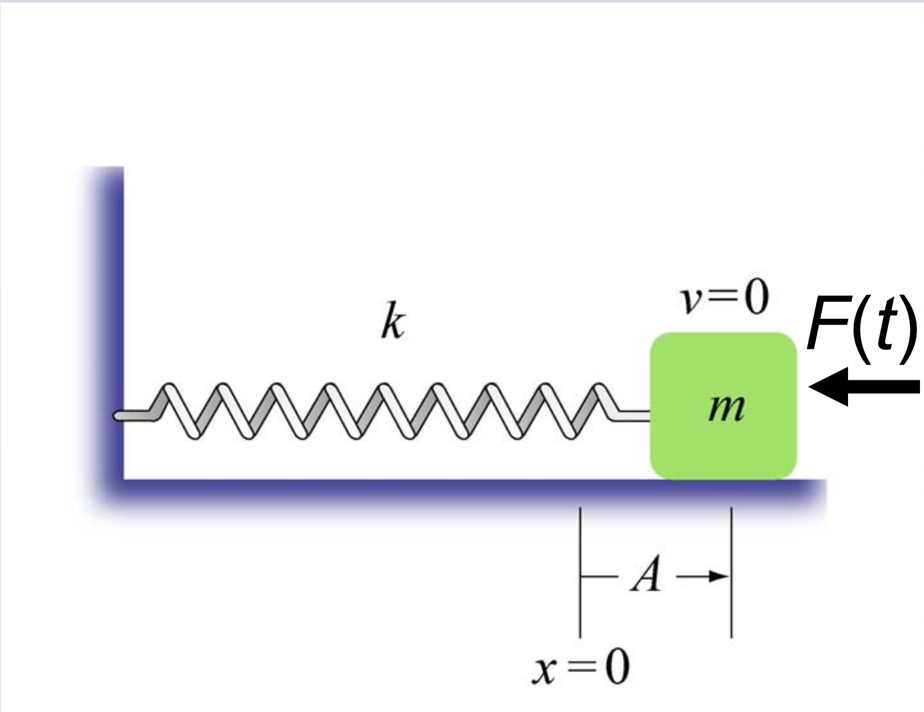
$$x(t) = x_0 \cos(\omega_0 t + \phi)$$

Moves at natural frequency

What if we now move the wall?
Push on the mass?

Demonstration:
Driven Mass on a Spring Off Resonance

Driven Mass on a Spring



Now we get:

$$F = F(t) - kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = F(t)$$

Assume harmonic force:

$$F(t) = F_0 \cos(\omega t)$$

Simple Harmonic Motion

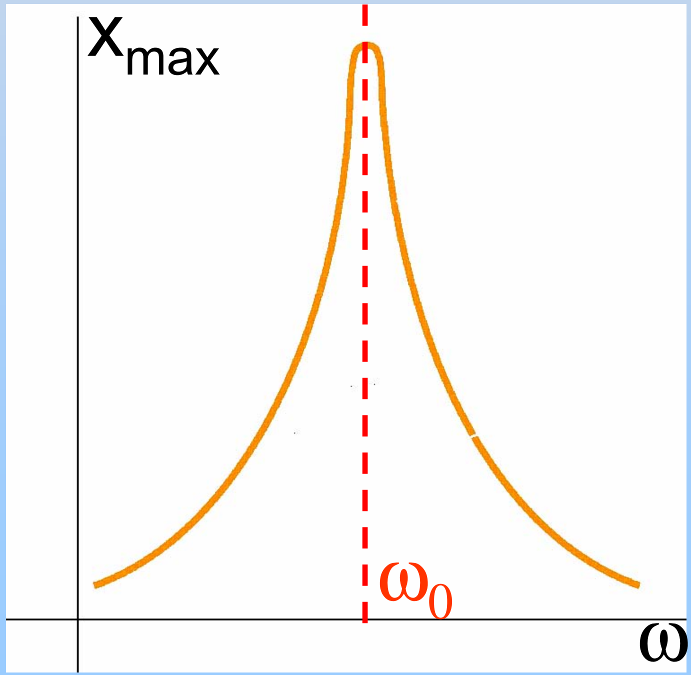
$$x(t) = x_{\max} \cos(\omega t + \phi)$$

Moves at driven frequency

Resonance

$$x(t) = x_{max}\cos(\omega t + \phi)$$

Now the amplitude, x_{max}, depends on how close the drive frequency is to the natural frequency



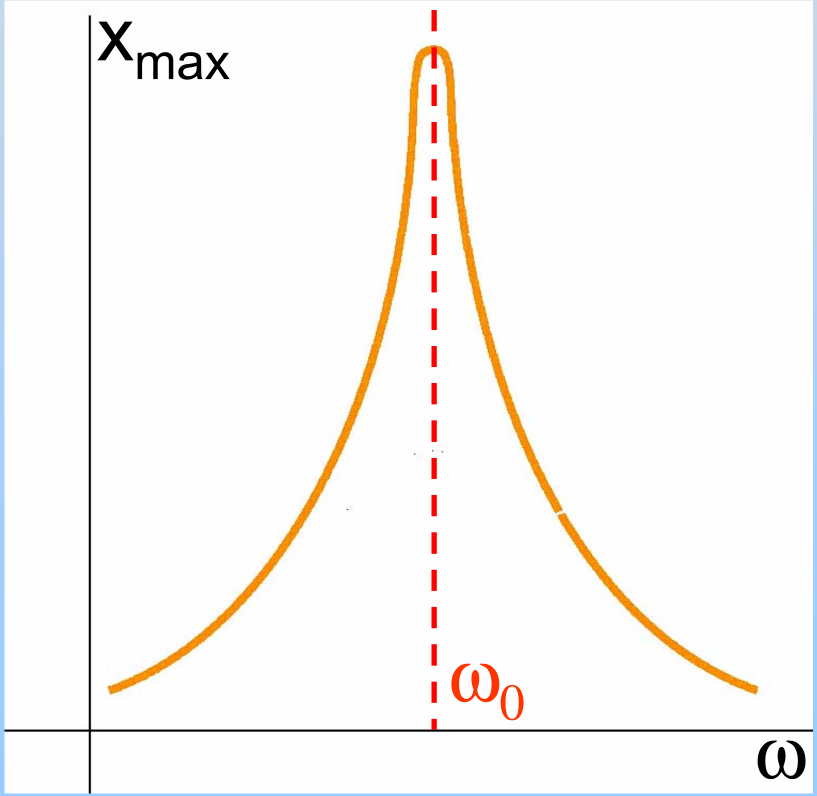
Let's See…

Demonstration: Driven Mass on a Spring

Resonance

$$x(t) = x_{\max} \cos(\omega t + \phi)$$

x_{max} depends on drive frequency



Many systems behave like this:

Swings

Some cars

Musical Instruments

…

Electronic Analog: RLC Circuits

Analog: RLC Circuit

Recall:

Inductors are like masses (have inertia)

Capacitors are like springs (store/release energy)

Batteries supply external force (EMF)

Charge on capacitor is like position,

Current is like velocity – watch them resonate

Now we move to "frequency dependent batteries:"

AC Power Supplies/AC Function Generators

Demonstration:
RLC with Light Bulb

Start at Beginning: AC Circuits

Alternating-Current Circuit

- direct current (dc) – current flows one way (battery)
- alternating current (ac) – current oscillates

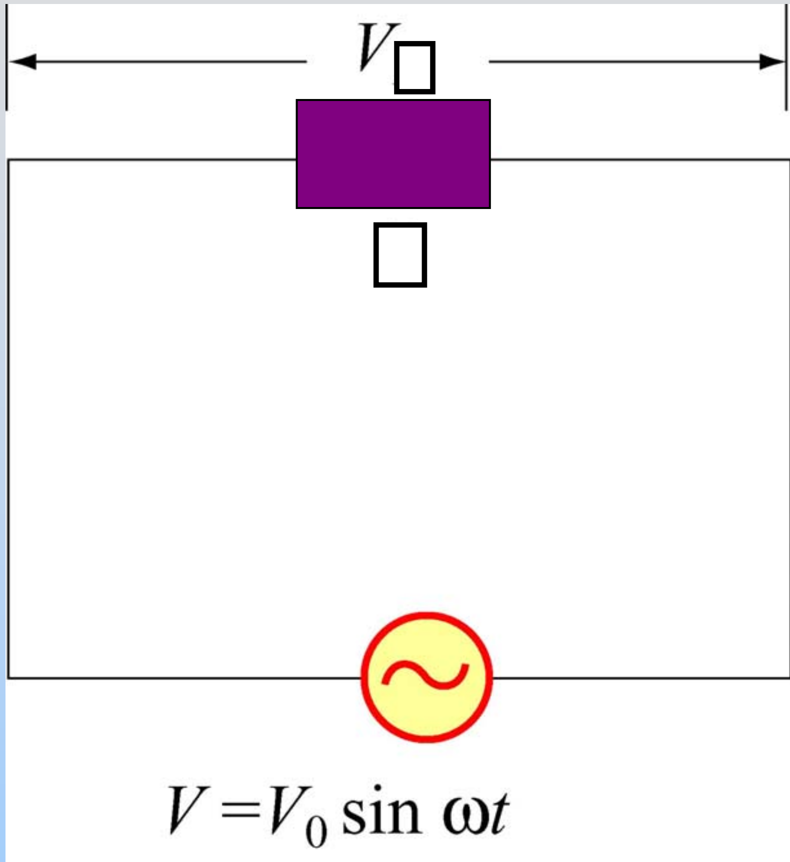
- sinusoidal voltage source

$$V(t) = V_0 \sin \omega t$$

$$\omega = 2\pi f : \text{ angular frequency}$$

$$V_0 : \text{ voltage amplitude}$$

AC Circuit: Single Element



$$V_{\square} = V$$

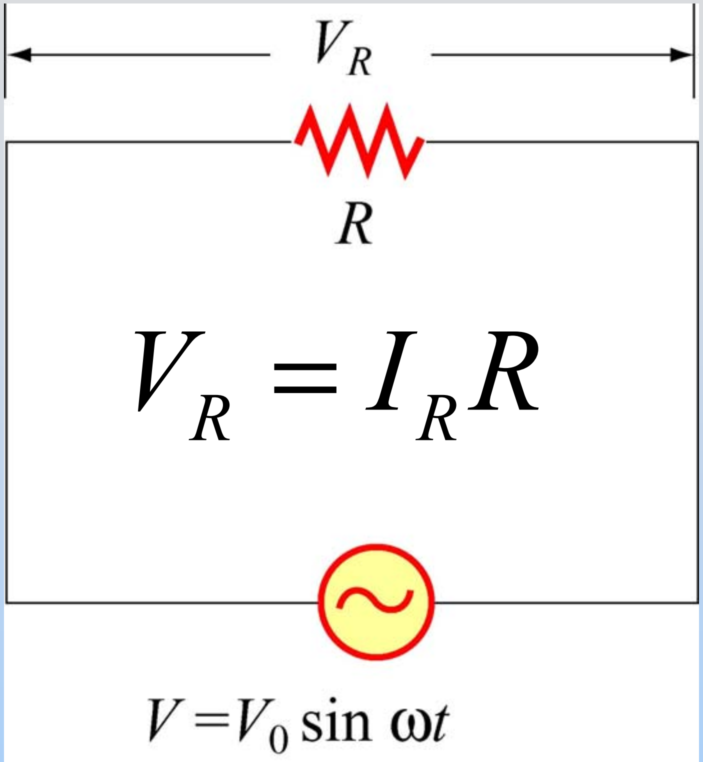
$$= V_0 \sin \omega t$$

$$I(t) = I_0 \sin(\omega t - \phi)$$

Questions:

1. What is I_0?
2. What is ϕ?

AC Circuit: Resistors



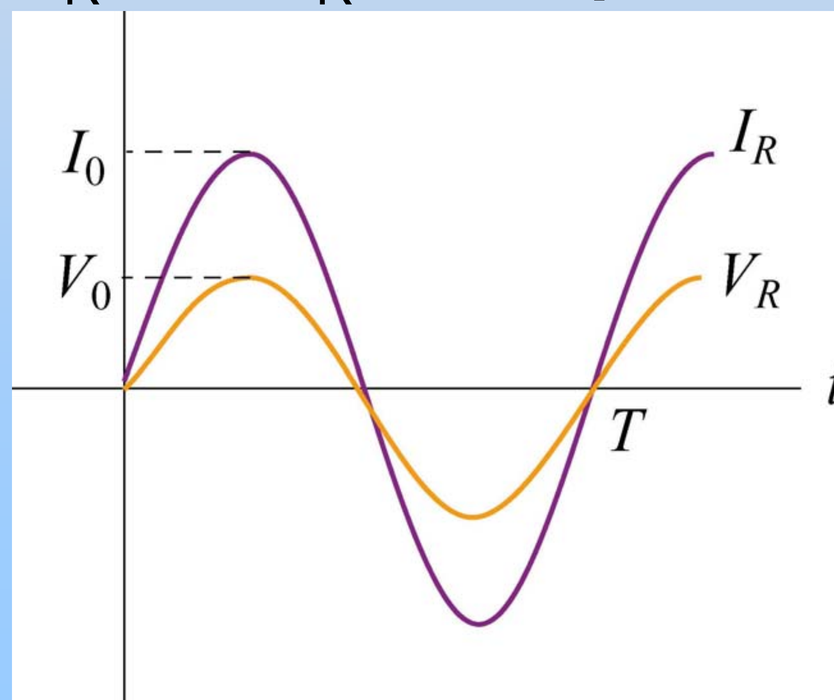
$$I_R = \frac{V_R}{R} = \frac{V_0}{R}\sin \omega t$$

$$= I_0 \sin\left(\omega t - 0\right)$$

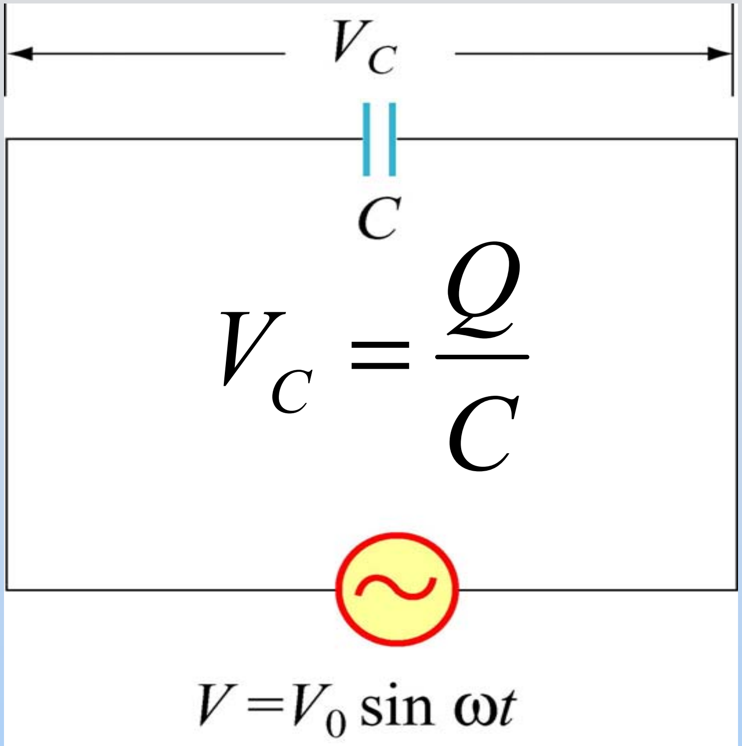
$$I_0 = \frac{V_0}{R}$$

$$\varphi = 0$$

I_R and V_R are **in phase**



AC Circuit: Capacitors



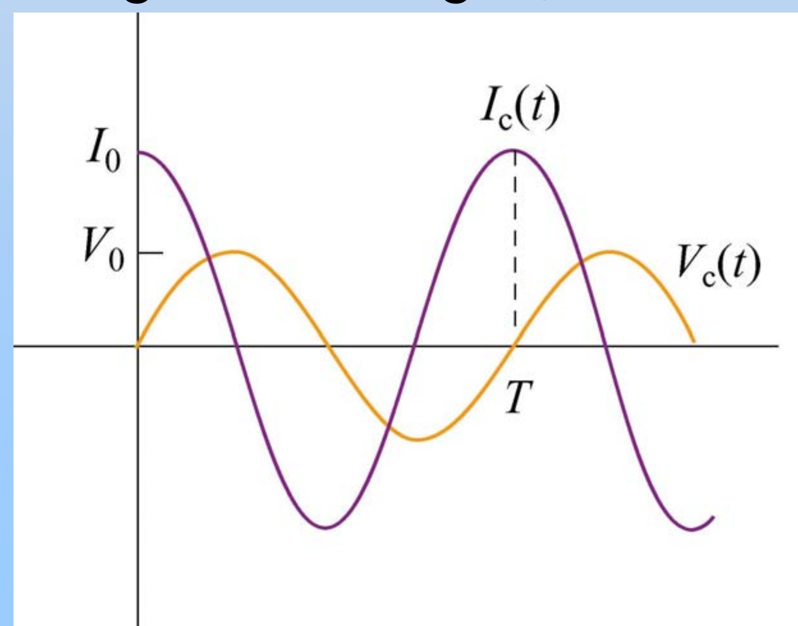
$$V_C = \frac{Q}{C}$$

$$Q(t) = CV_C = CV_0 \sin \omega t$$

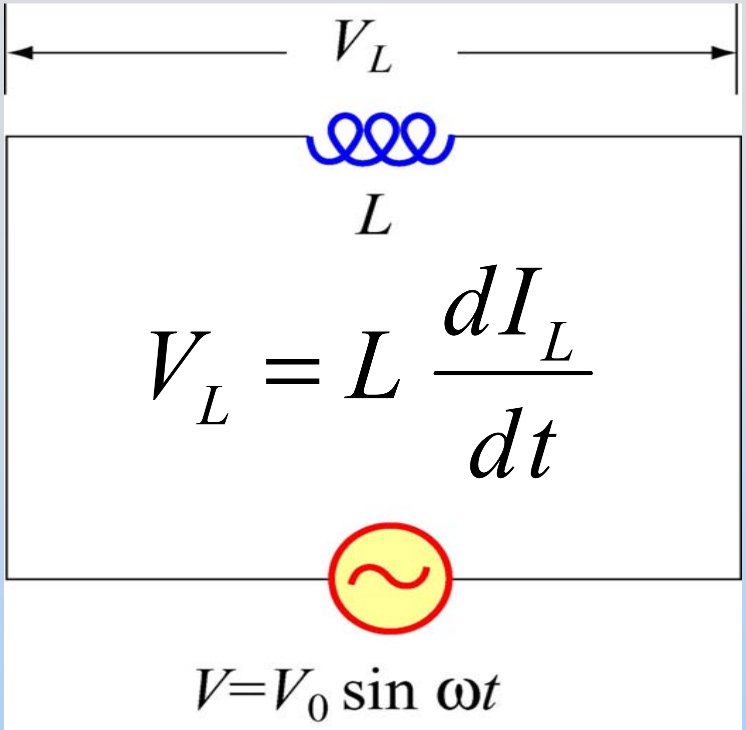
$$I_C(t) = \frac{dQ}{dt}$$
$$= \omega C V_0 \cos \omega t$$
$$= I_0 \sin(\omega t - {}^{-\pi}/_{2})$$

$$I_0 = \omega C V_0$$
$$\varphi = -\pi/2$$

I_C leads V_C by $\pi/2$



AC Circuit: Inductors



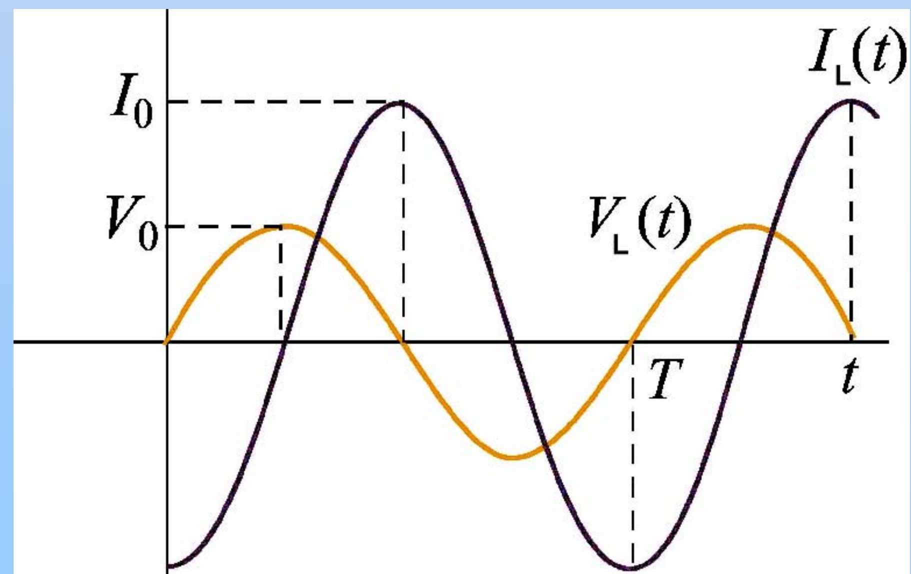
$$V_L = L\frac{dI_L}{dt}$$

$$I_L(t) = \frac{V_0}{L}\int \sin \omega t \; dt$$
$$= -\frac{V_0}{\omega L}\cos \omega t$$
$$= I_0 \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$I_0 = \frac{V_0}{\omega L}$$
$$\varphi = \frac{\pi}{2}$$

$$\frac{dI_L}{dt} = \frac{V_L}{L} = \frac{V_0}{L}\sin \omega t$$

I_L **lags** V_L by π/2



AC Circuits: Summary

Element	I_0	Current vs. Voltage	Resistance Reactance Impedance
Resistor	$\frac{V_{0R}}{R}$	In Phase	$R = R$
Capacitor	$\omega C V_{0C}$	Leads	$X_C = \frac{1}{\omega C}$
Inductor	$\frac{V_{0L}}{\omega L}$	Lags	$X_L = \omega L$

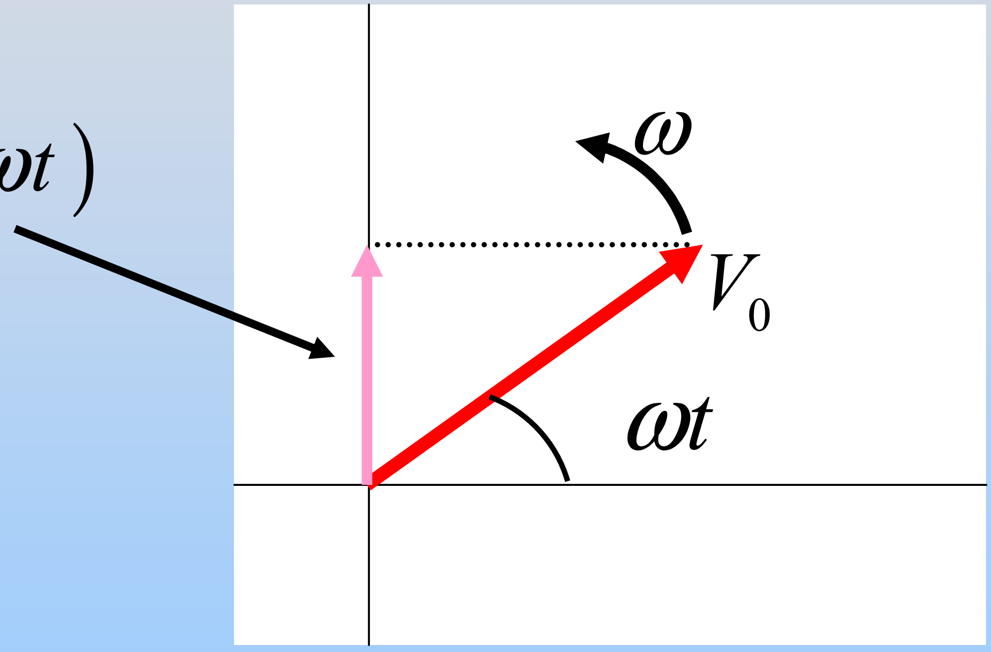
Although derived from single element circuits, these relationships hold generally!

PRS Question:
Leading or Lagging?

Phasor Diagram

Nice way of tracking magnitude & phase:

$$V(t) = V_0 \sin(\omega t)$$

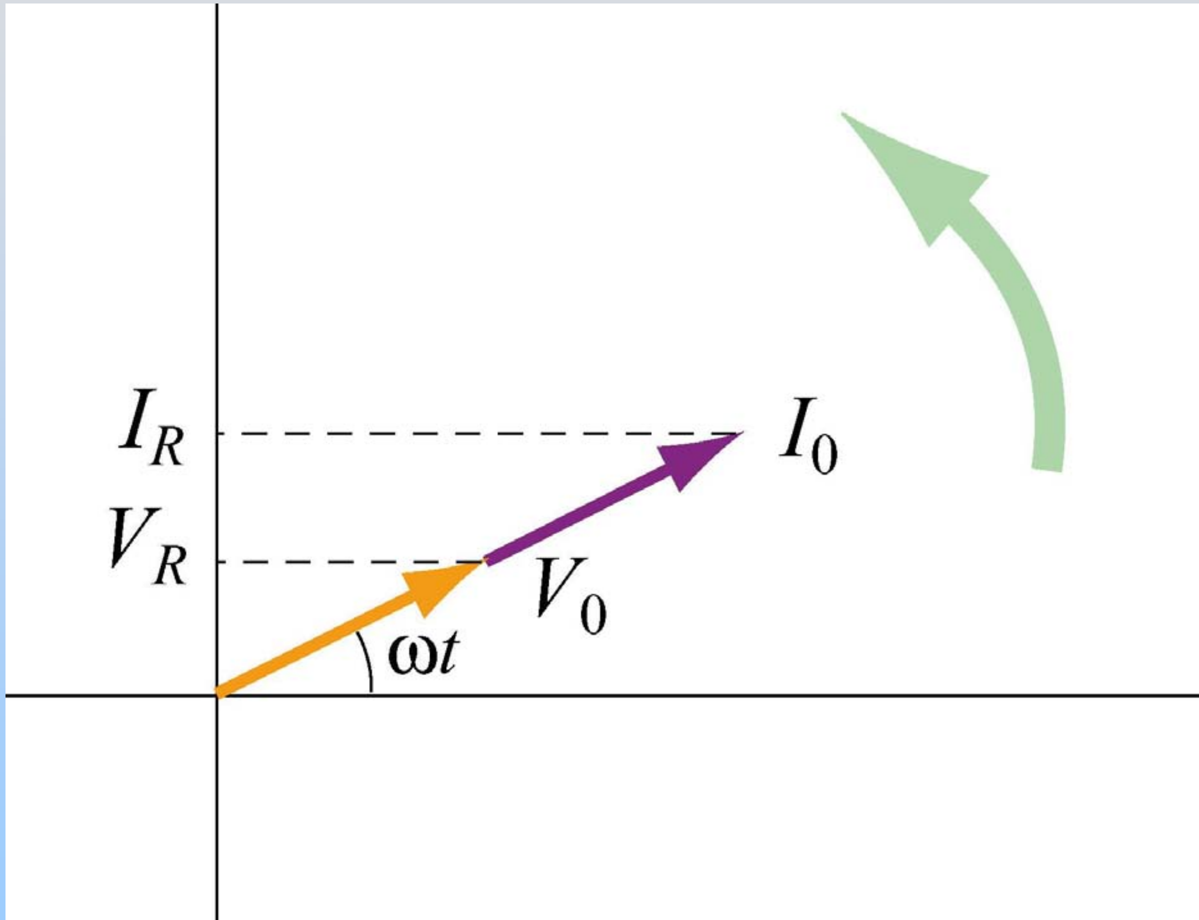


Notes: (1) As the phasor (red vector) rotates, the projection (pink vector) oscillates

(2) Do both for the current and the voltage

Demonstration: Phasors

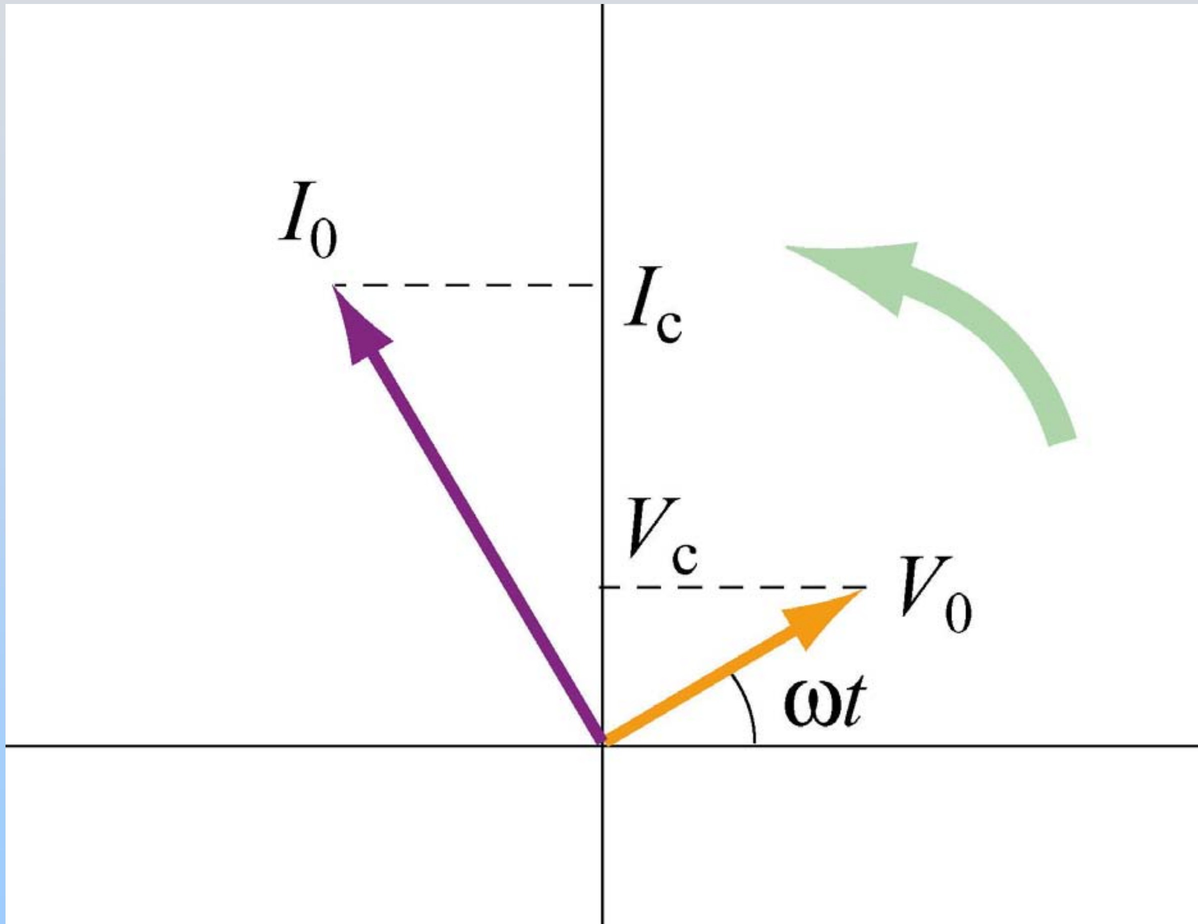
Phasor Diagram: Resistor



$$V_0 = I_0 R$$
$$\varphi = 0$$

I_R and V_R are **in phase**

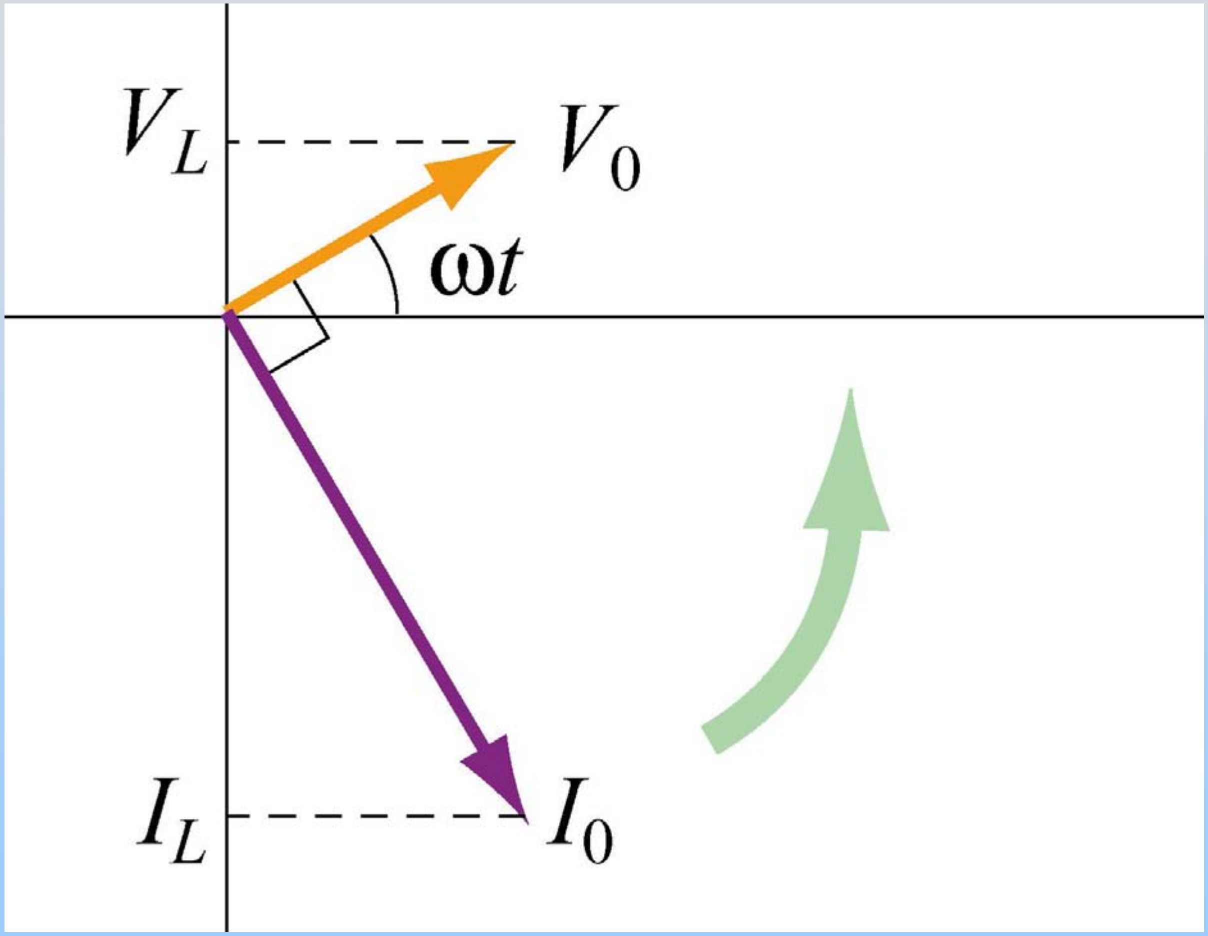
Phasor Diagram: Capacitor



$$V_0 = I_0 X_C$$
$$= I_0 \frac{1}{\omega C}$$
$$\varphi = -\pi/2$$

I_C **leads** V_C by $\pi/2$

Phasor Diagram: Inductor



$$V_0 = I_0 X_L$$
$$= I_0 \omega L$$
$$\varphi = \frac{\pi}{2}$$

I_L **lags** V_L by $\pi/2$

PRS Questions: Phase

Put it all together:
Driven RLC Circuits

Question of Phase

We had fixed phase of voltage:

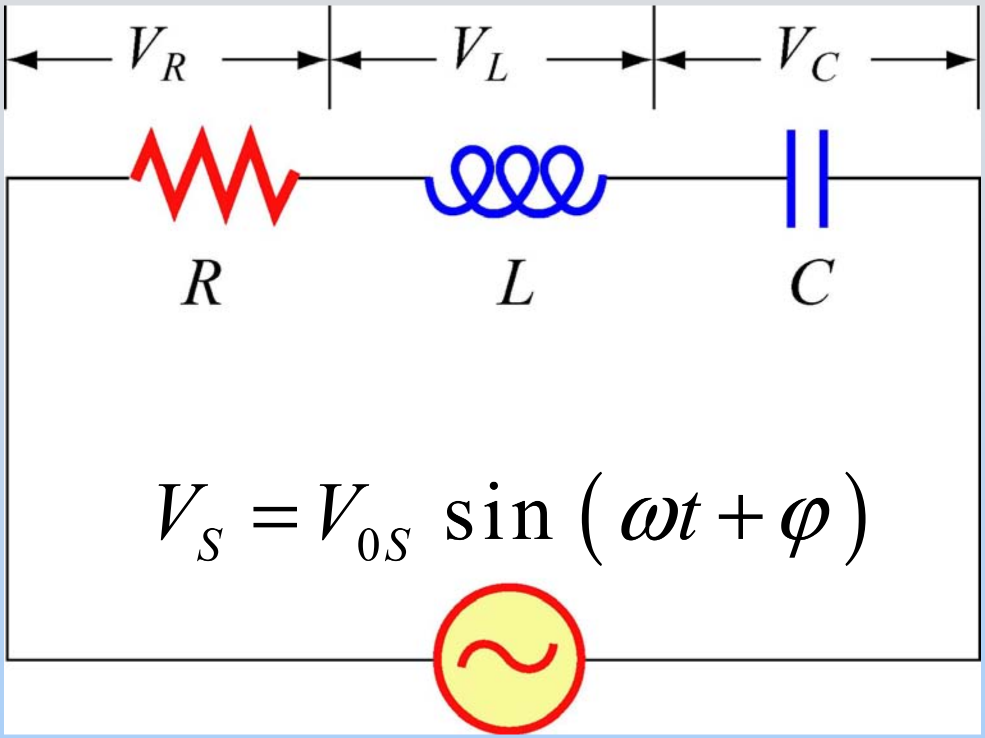
$$V = V_0 \sin \omega t \qquad I(t) = I_0 \sin(\omega t - \phi)$$

It's the same to write:

$$V = V_0 \sin(\omega t + \phi) \quad I(t) = I_0 \sin \omega t$$

(Just shifting zero of time)

Driven RLC Series Circuit



$$I(t) = I_0 \sin(\omega t)$$

$$V_R = V_{R0} \sin(\omega t)$$

$$V_L = V_{L0} \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

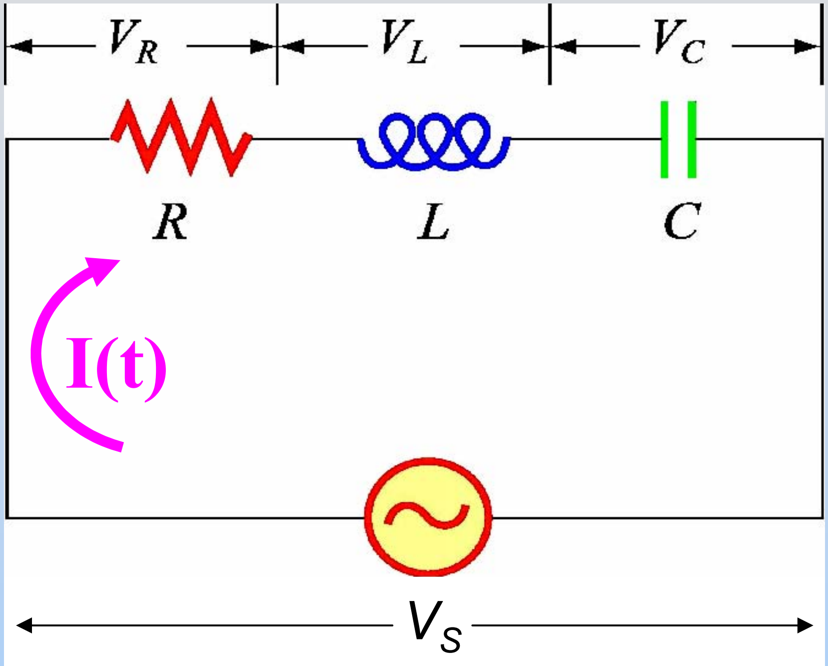
$$V_C = V_{C0} \sin\left(\omega t + \tfrac{-\pi}{2}\right)$$

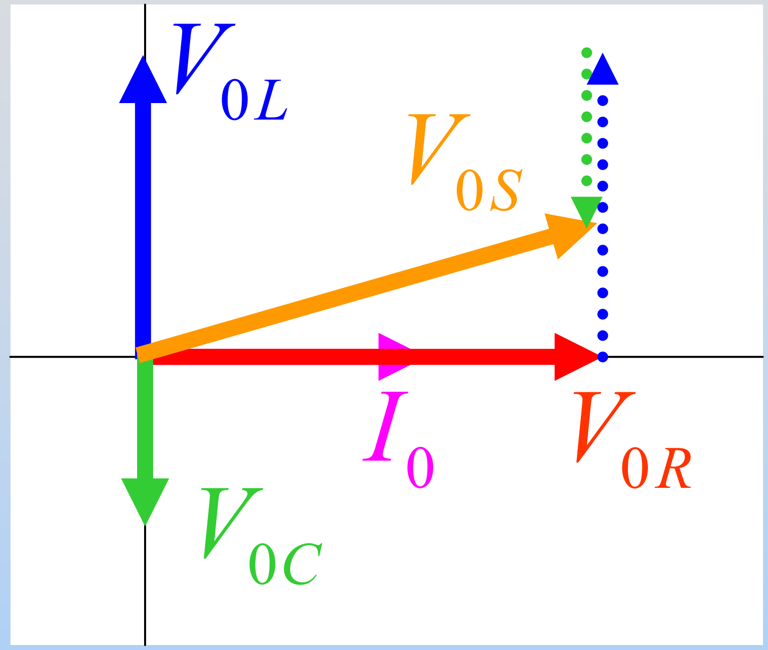
What is I_0 (and $V_{R0} = I_0 R$, $V_{L0} = I_0 X_L$, $V_{C0} = I_0 X_C$)?

What is φ? Does the current lead or lag V_s?

Must Solve: $V_S = V_R + V_L + V_C$

Driven RLC Series Circuit

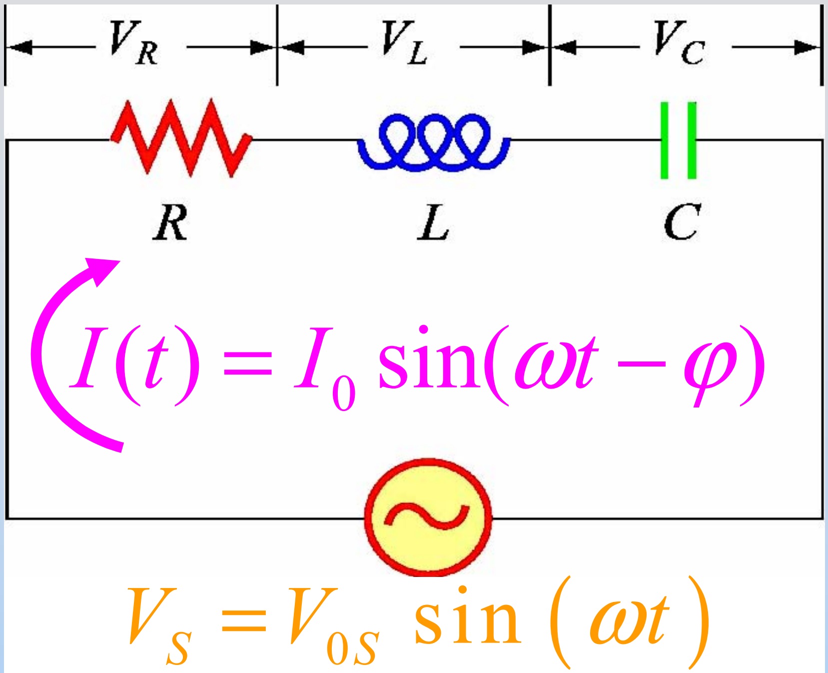


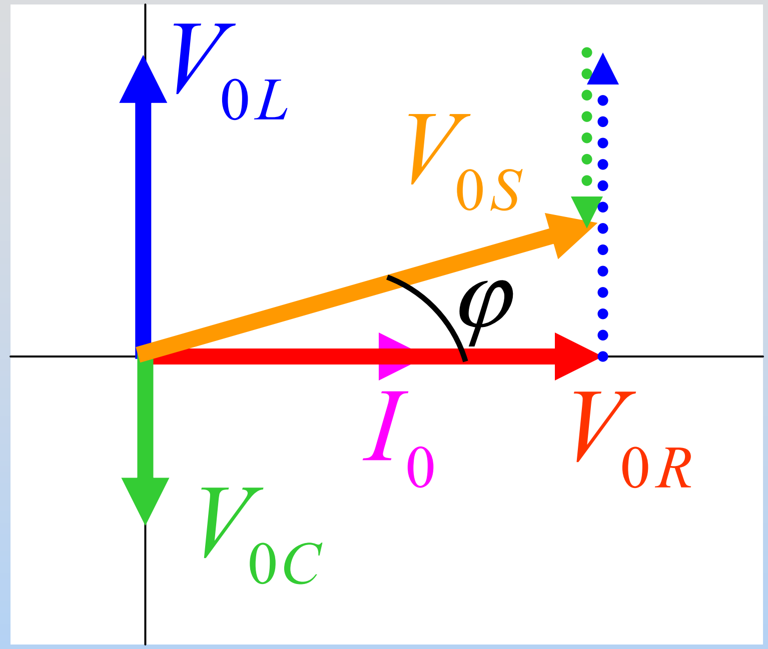


Now Solve: $V_S = V_R + V_L + V_C$

Now we just need to read the phasor diagram!

Driven RLC Series Circuit





$$V_{0S} = \sqrt{V_{R0}^{\;2} + (V_{L0} - V_{C0})^2} = I_0\sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z$$

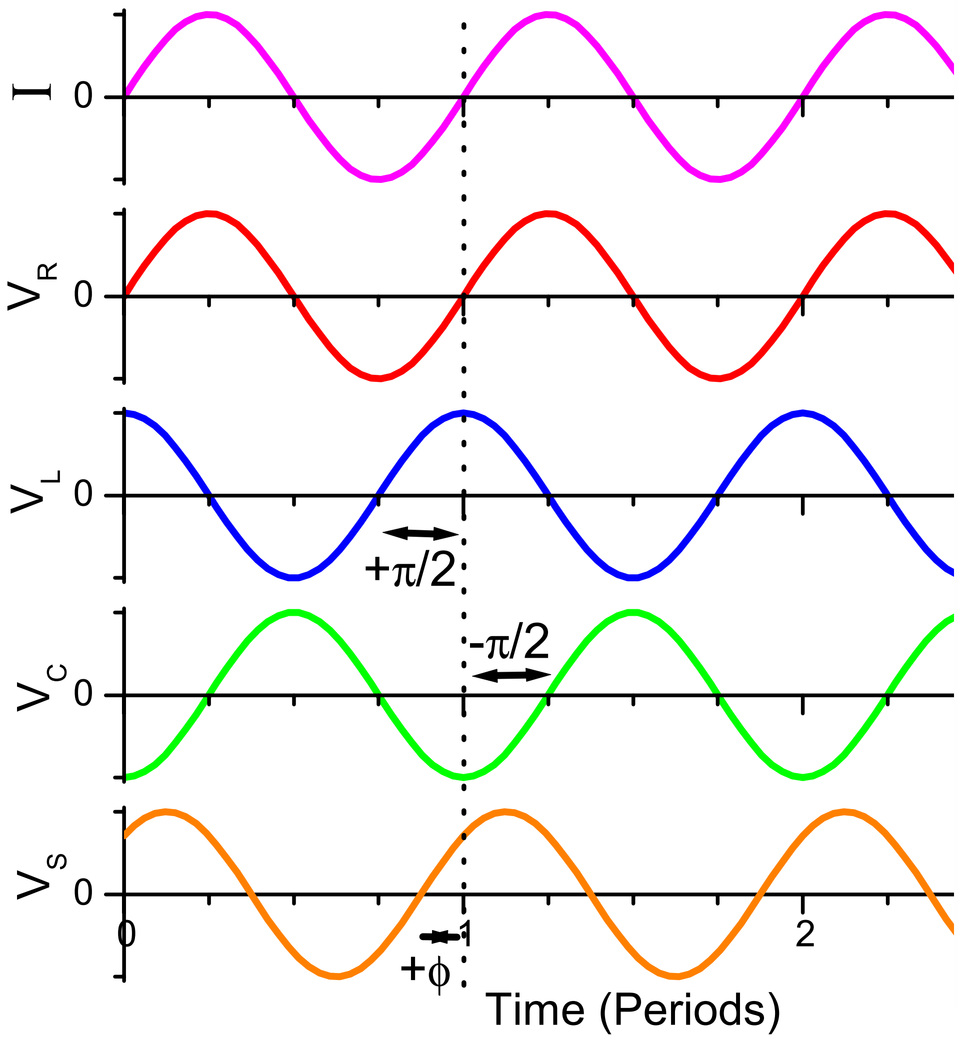
$$I_0 = \frac{V_{0S}}{Z}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Plot I, V's vs. Time



$$I(t) = I_0 \sin(\omega t)$$

$$V_R(t) = I_0 R \sin(\omega t)$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

$$V_L(t) = I_0 X_C \sin\left(\omega t - \tfrac{\pi}{2}\right)$$

$$V_S(t) = V_{S0} \sin(\omega t + \varphi)$$

PRS Question:
Who Dominates?

RLC Circuits: Resonances

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

At very low frequencies, C dominates ($X_C >> X_L$):
it fills up and keeps the current low

At very high frequencies, L dominates ($X_L >> X_C$):
the current tries to change but it won't let it

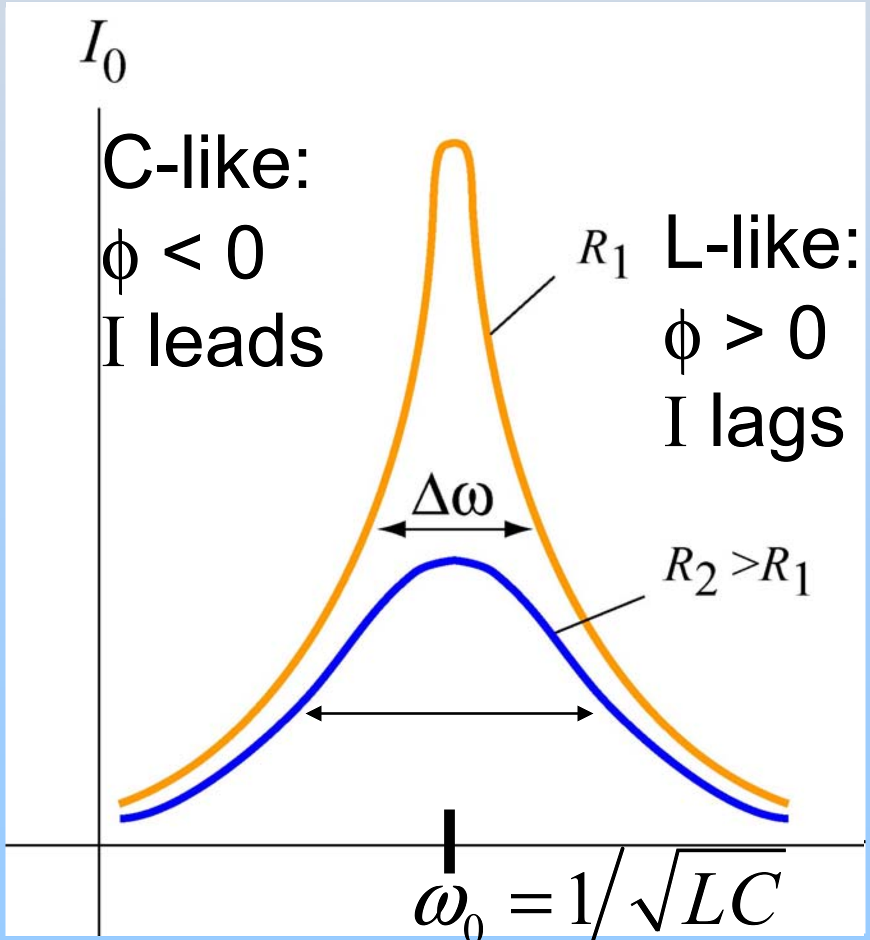
At intermediate frequencies we have **resonance**

I_0 reaches maximum when $X_L = X_C$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}\ ; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$



$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Demonstration:
RLC with Light Bulb

PRS Questions: Resonance

Experiment 11: Driven RLC Circuit

Experiment 11: How To

Part I

- Use exp11a.ds
- Change frequency, look at I & V. Try to find resonance – place where I is maximum

Part II

- Use exp11b.ds
- Run the program at each of the listed frequencies to make a plot of I_0 vs. ω