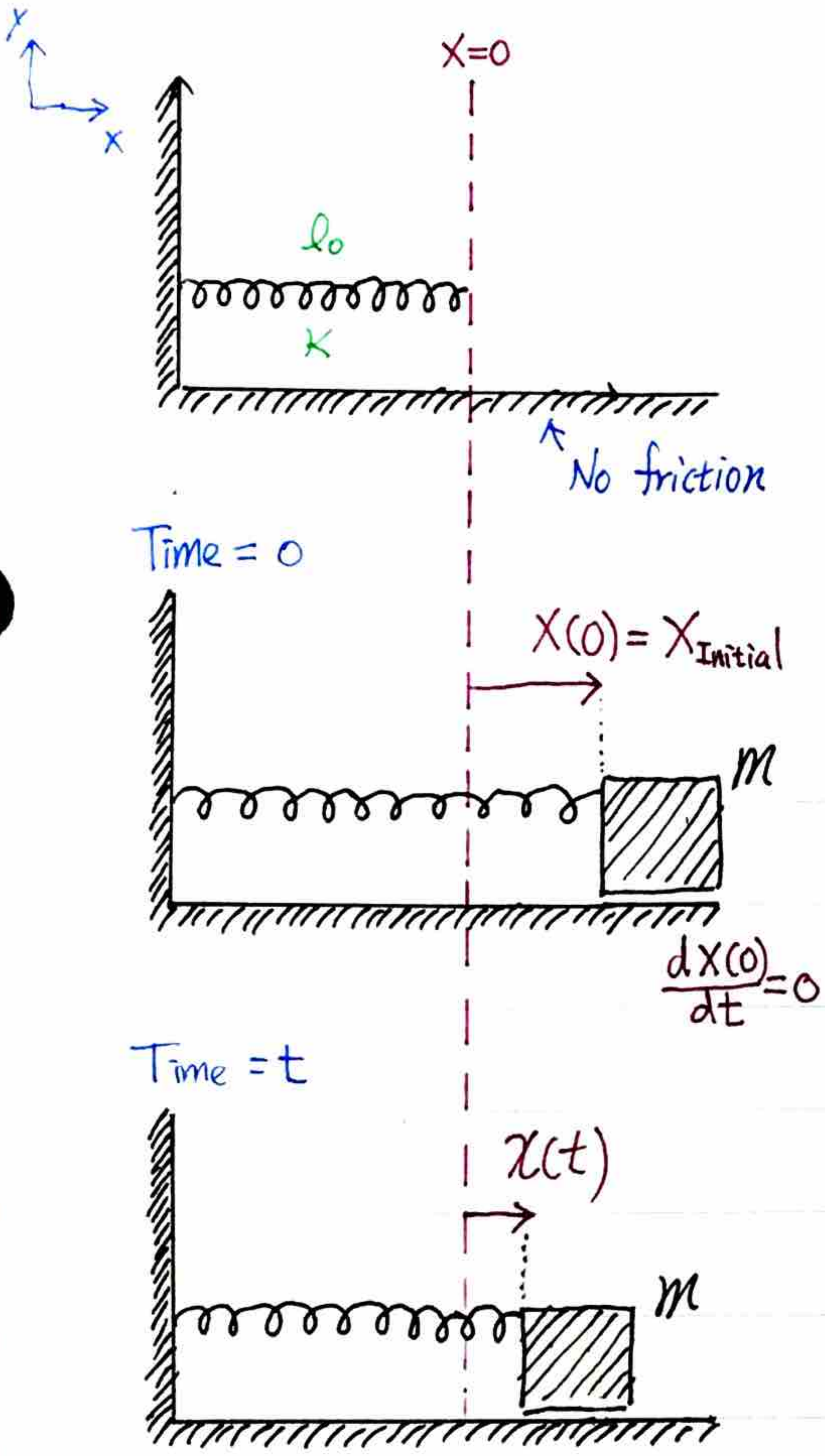
We are trying to understand the motion of a mass attached to an idealized spring!




Spring Constant $= K$

Spring natural length $= l_0$

Define coordinate system $x$

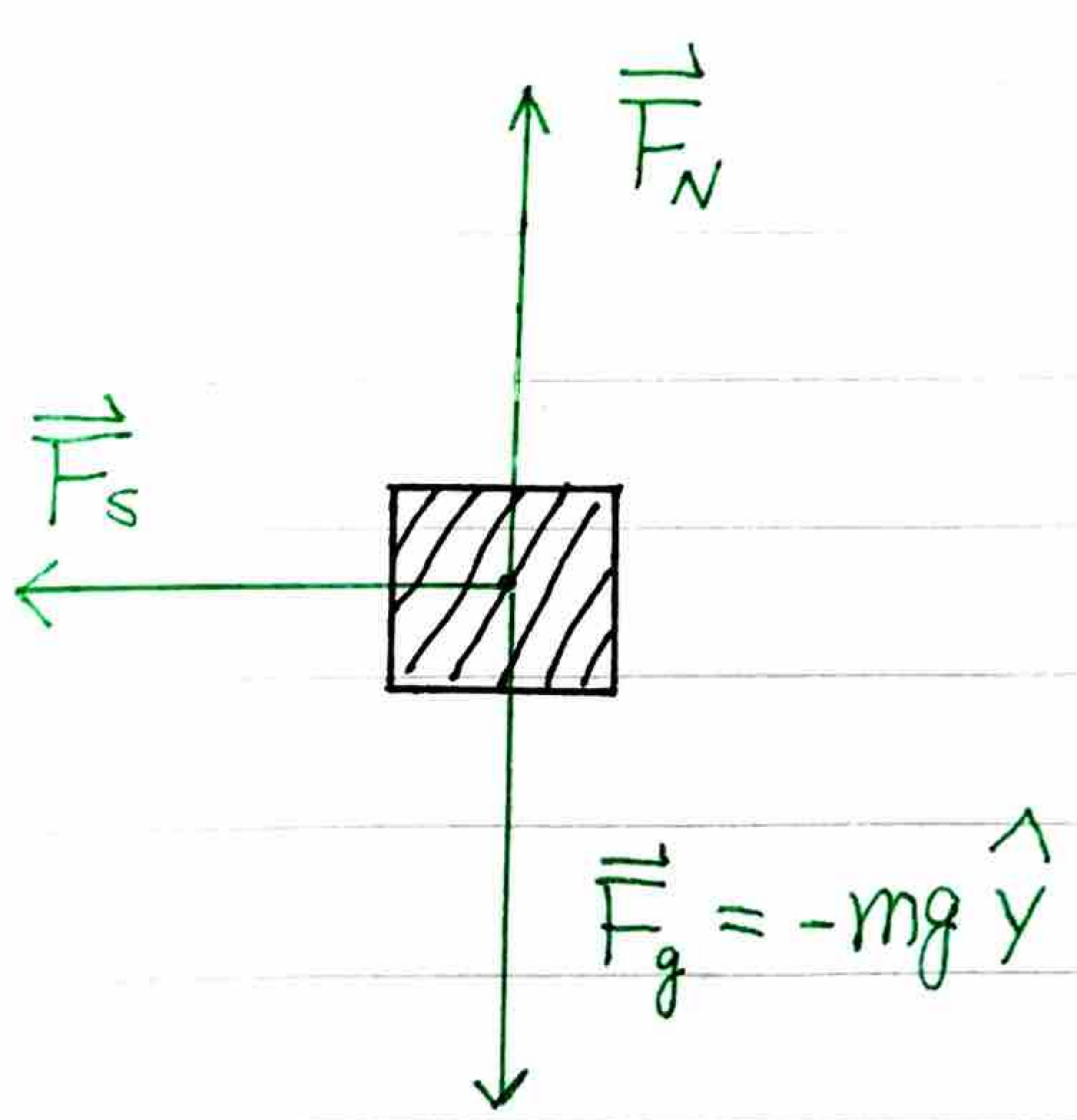
$x=0$ : "Equilibrium Position"

$m$: moves only in the $x$ direction

Goal: predict what will happen at time $= t$

* How many forces are acting on the block?

Force Diagram:




Total force : $\vec{F}$

$$\vec{F} = \vec{F}_s + \vec{F}_N + \vec{F}_g$$

We know that the mass only move in the $x$ direction

$$\vec{F}_N = -\vec{F}_g = mg\,\hat{y}$$

$$\Rightarrow \vec{F} = \vec{F}_s = -K\,x(t)\,\hat{x}$$

* From Newton's Law

$$\vec{F} = m\vec{a} = m\frac{d^2x(t)}{dt^2}\hat{x} = m\ddot{x}(t)\hat{x}$$

$$= -Kx(t)\hat{x}$$ (from force diagram)

Since everything is in $x$ direction (drop $\hat{x}$)

$$\ddot{X} = -\frac{K}{m}x = -\omega^2 x$$

$$\left(\omega \equiv \sqrt{\frac{K}{m}}\right)$$ to make life easier :)

Now we have successfully translated a physical situation into a mathematical description :

⇒ I have the equation of motion

I have the initial conditions

Solution : $x(t) = a\cos\omega t + b\sin\omega t$

($a$ and $b$ are arbitrary)

This equation satisfies my equation !! 2 unknown !

From "Uniqueness Theorem"

⇒ This is the one and only one solution in our universe which satisfies the equation !

Use the initial condition :

$$\begin{cases} ① \; x(0) = x_{Initial} \\ ② \; \dot{x}(0) = 0 \end{cases}$$

① $\Rightarrow \quad a = x_{Initial}$

② $\Rightarrow \quad 0 = b\omega \quad \Rightarrow b = 0$

Finally : $\quad x(t) = \underset{\substack{\Uparrow \\ \text{amplitude}}}{x_{Initial}} \; \underset{\substack{\Uparrow \\ \text{Harmonic Oscillation}}}{\cos(\omega t)}$

$$\omega = \sqrt{\frac{K}{m}}$$

Let's stop for a second and consider what we have done :

(1) Take a physical situation and translate it to a mathematical description

(2) Solve the equation

(3) The solution actually matches what the nature do to the mass

This is amazing !

DEMO

* Nobody understand why the nature can be described by mathematics ...

This means that we use "the same tool" for the prediction of Higgs Boson, Quark Gluon Plasma, Gravitation Waves and the motion of the mass in this example (!?)

* Quote from Einstein :

"The most incomprehensible thing about the universe is that it is comprehensible"

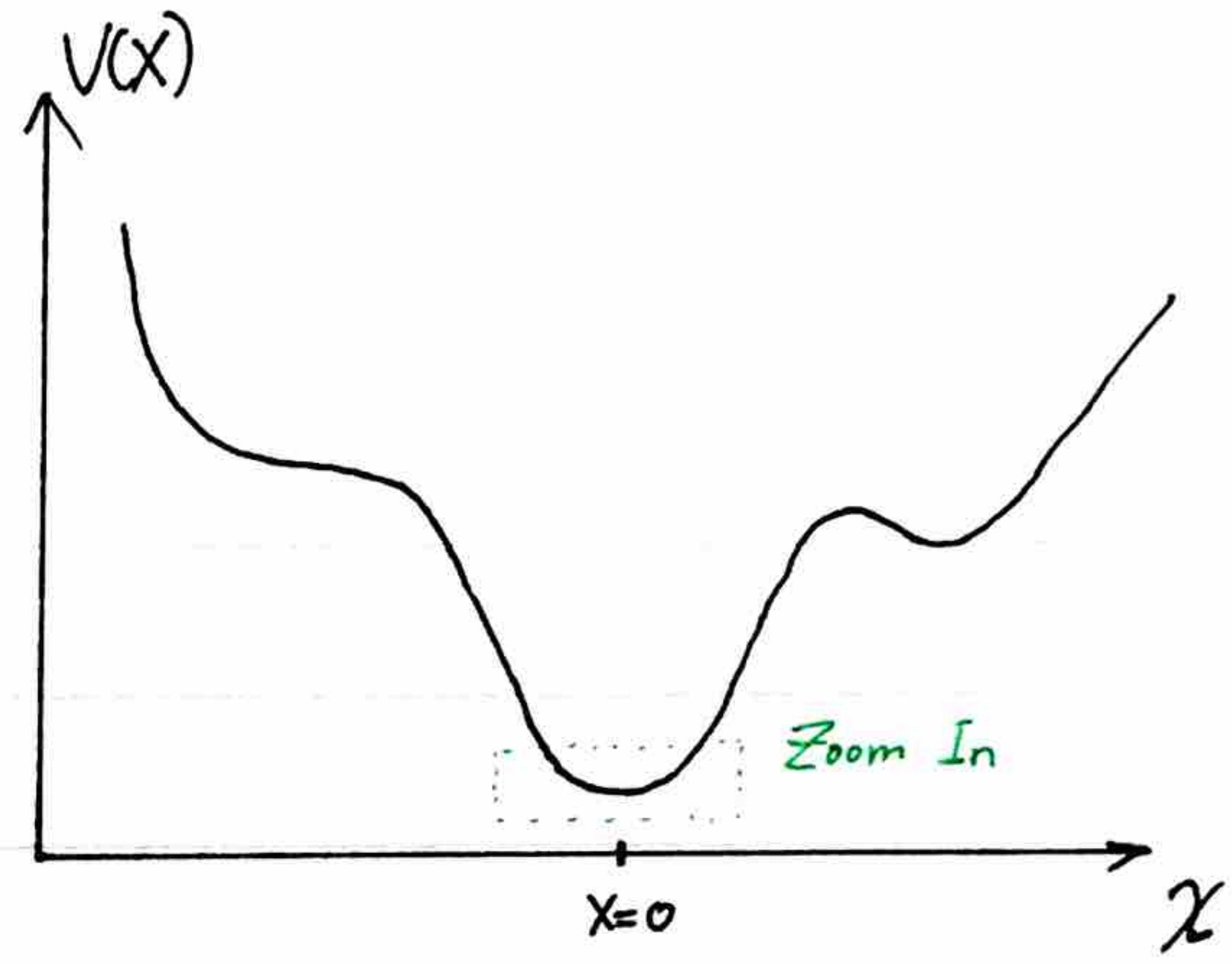
* Quote from Rene Descartes :

"But in my opinion, everything in nature occurs mathematically"

We have just solved a problem with "ideal" spring which follows Hooke's Law. What is so special about it?

Actually, there is NO "Hooke's Law"! Breaks down at some point. But the law is a very good approximation when we consider small amplitude vibration

Consider a potential $V(x)$




at $x=0$

$\Downarrow$

$V(0)$: minimum "equilibrium position"

$\Updownarrow$

$$F(0) = -\frac{d}{dx}V(x)\Big|_{x=0} = -V'(0) = 0$$

Consider small oscillation about equilibrium position:

Taylor's Expension: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$$\Rightarrow V(x) = V(0) + \frac{V'(0)}{1!}x + \frac{V''(0)}{2!}x^2 + \frac{V'''(0)}{3!}x^3 + \cdots$$

$$F(x) = -\frac{d}{dx}V(x) = -V'(0) - V''(0)x - \frac{1}{2}V'''(0)x^2 + \cdots$$

Given $V'(0)=0$; when $x$ is small enough:

$$F(x) \approx -V''(0)x$$

This is a remarkable result:

"Hooke's Law" works on "all systems with a smooth potential" (also $V''(0) \neq 0$) for small oscillations about stable equilibrium!

How small?

Ans: $$|x V'''(0)| \ll V''(0)$$

* We have solved all those kind of situations!

* On the other hand, when $x \to$ large $\Rightarrow$ Non-linear term (ex: $V'''(0)$ term) become more and more important.

* In 8.03, we focus on linear systems.

Come back to this equation of motion (E.O.M)

$$\ddot{x} + \omega^2 x = 0$$

There are two important properties of this linear E.O.M

(1) If $x_1(t)$ and $x_2(t)$ are both solutions

$\Rightarrow$ $x_{12}(t) = x_1(t) + x_2(t)$ is also a solution!

(2) Time translation invariance:

If $x(t)$ is a solution $\Rightarrow$ $x(t') = x(t+a)$ is also a solution

This is because of the chain rule:

$$\frac{d}{dt} x(t+a) = \frac{d(t+a)}{dt} \left. \frac{dx(t')}{dt'} \right|_{t'=t+a} = \left. \frac{dx(t')}{dt'} \right|_{t'=t+a}$$

This means that if I change $t=0$ $\Rightarrow$ the physics will be the same!

Most of the physics systems are time translation invariant in the absence of an external force

To break the symmetry: need to make $K$ (thus $\omega$) time depent!

break

Solution to $\ddot{x} + \omega^2 x = 0$

① $x(t) = a \cos\omega t + b \sin\omega t$

$a$ & $b$ are arbitrary constants

We can write it in different forms!

② $x(t) = A \cos(\omega t + \phi)$

$A$ & $\phi$ are arbitrary constants

$= (A\cos\phi)\cos\omega t - (A\sin\phi)\sin\omega t$

③ $x(t) = \mathrm{Re}\left[A e^{i(\omega t + \phi)}\right]$ $\qquad i = \sqrt{-1}$

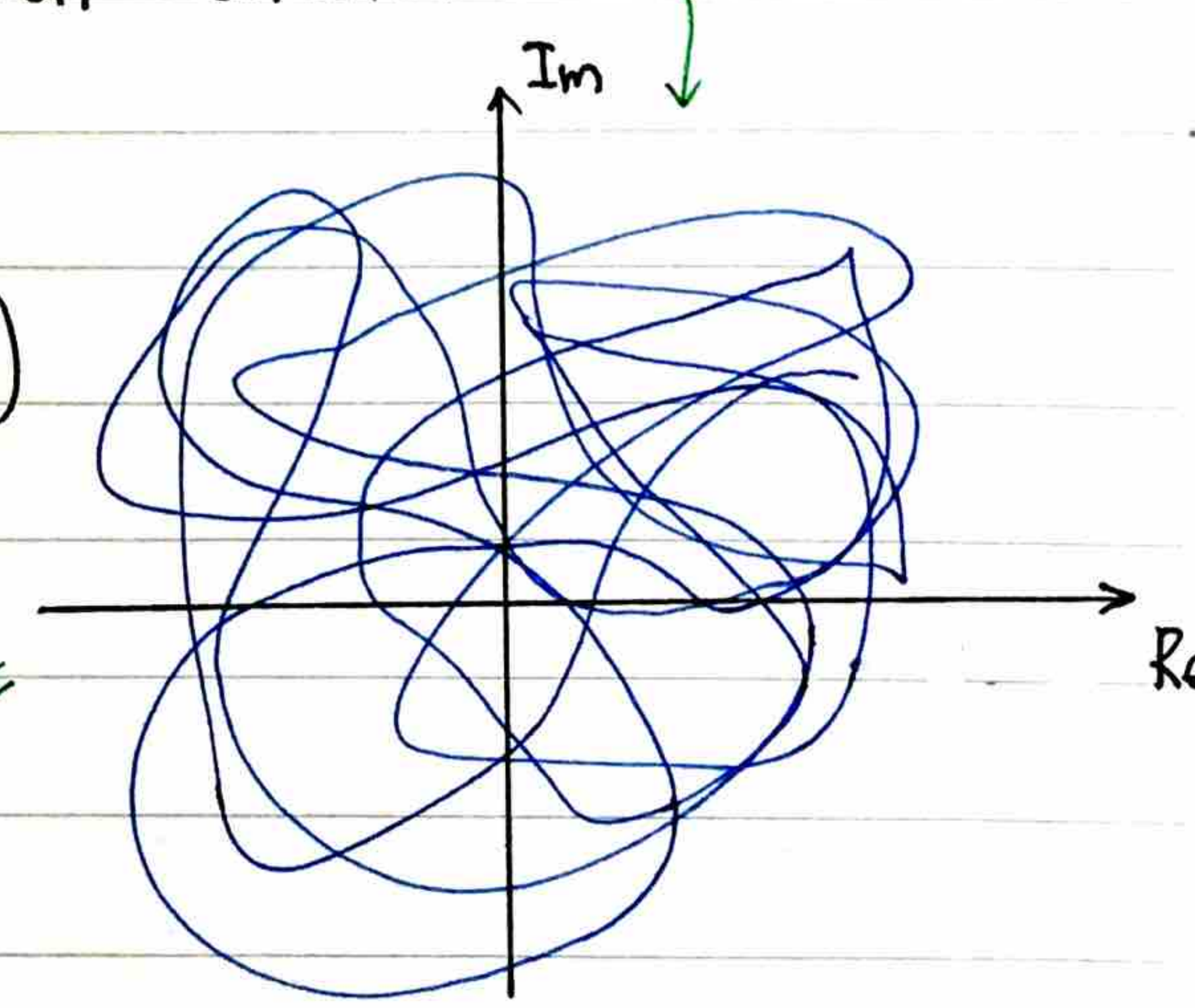
①~③ are the same solution, but written in different forms.

③ is a mathematical trick. In principle

$x(t) = \mathrm{Re}\left(A\cos(\omega t + \phi) + i f(t)\right)$

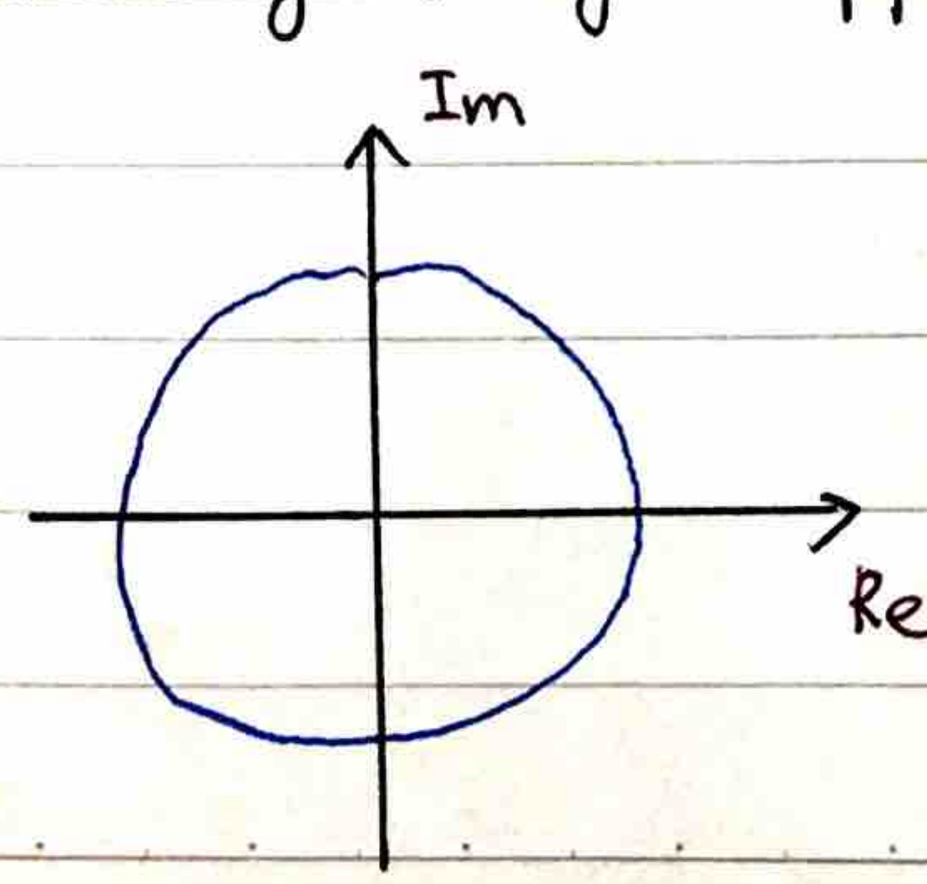
$f(t)$ is an arbitrary real function

if we plot it:




But if $f(t) = A\sin(\omega t + \phi)$

amazing thing happens!



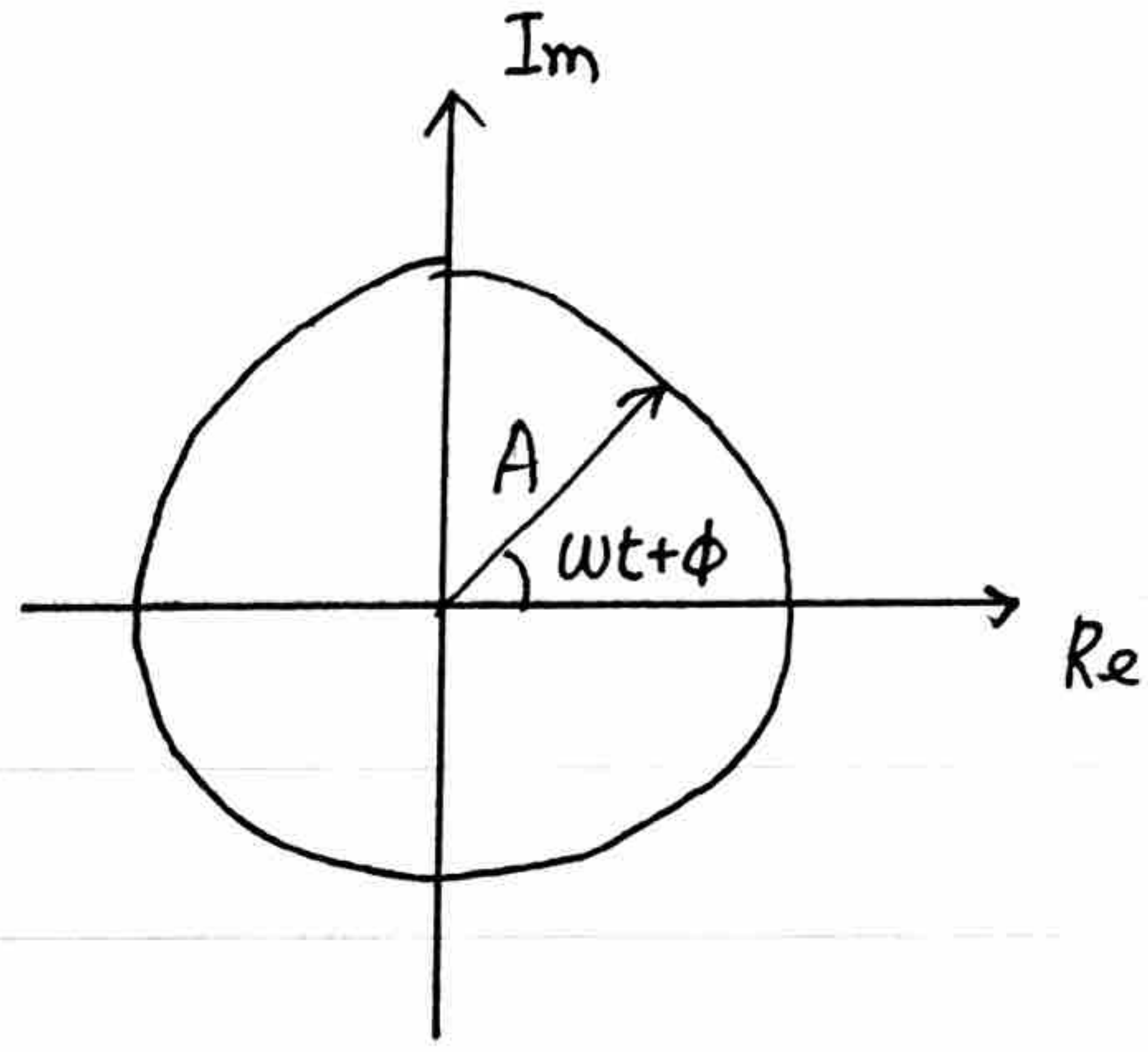

A circle!!!

$$X(t) = Re\left(A\cos(\omega t+\phi) + iA\sin(\omega t+\phi)\right)$$
$$= Re\left(Ae^{i(\omega t+\phi)}\right)$$

note: $e^{i\theta} = \cos\theta + i\sin\theta$

What does this mean ?

"Phoenix function"




① Can not be killed by differentiation !!

② Have a very nice property :

$$e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

$$Ae^{i(\omega t+\phi)} \xrightarrow[\text{Time translation}]{t\to t+a} Ae^{i(\omega(t+a)+\phi)}$$

Time translation is just a rotation in the complex plane !