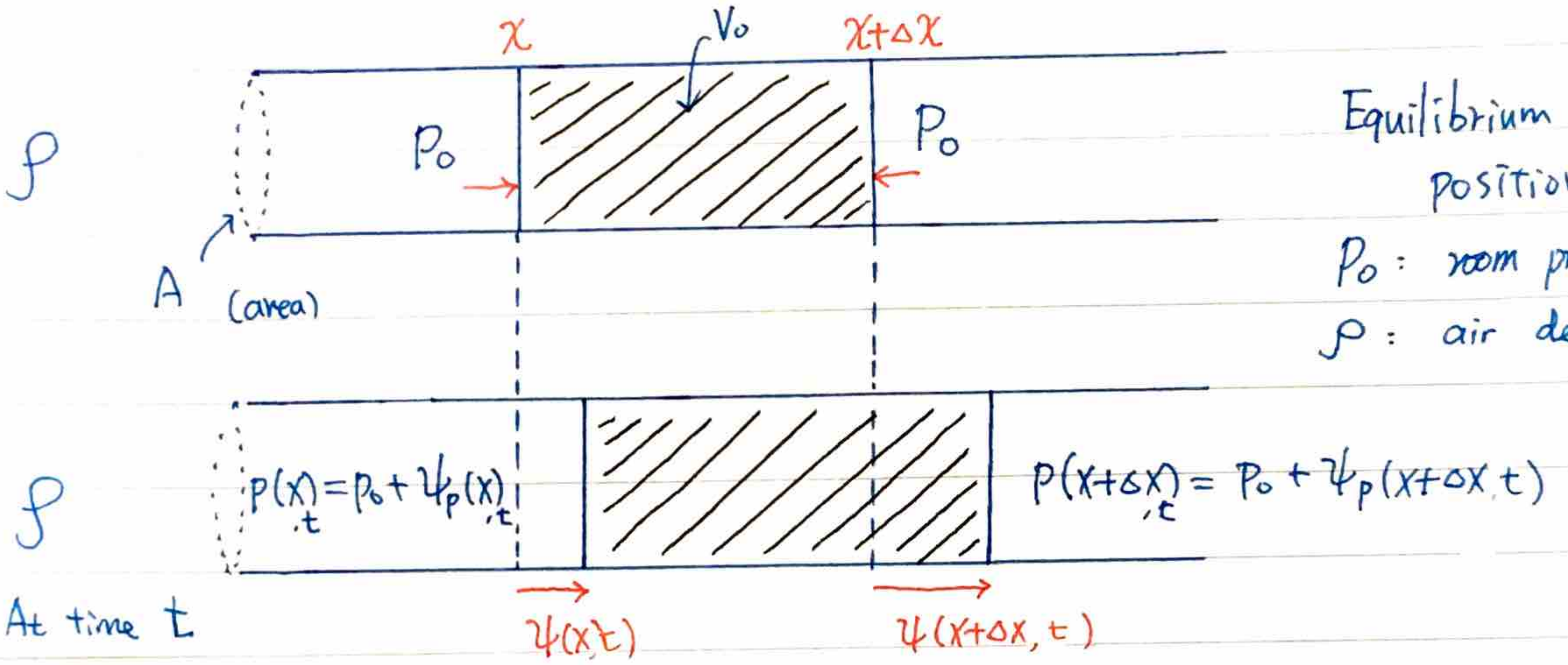
We have discussed the motion of a massive string extensively.

This time: more examples which can be described by the wave equation. (slide 1~8)

"Longitudinal waves"



Equilibrium position

P_0 : room pressure

ρ : air density.

Change in volume (ΔV): $A\left(\psi(x+\Delta x, t) - \psi(x,t)\right) \approx A\dfrac{\partial \psi}{\partial x}\Delta x$

Pressure difference (ΔP): $-\psi_p(x+\Delta x, t) + \psi_p(x,t) \approx -\dfrac{\partial \psi_p}{\partial x}\Delta x$

* Question: How do we relate pressure and volume?

Ideal Gas Law (slide 9-10)

(1) $pV = nRT \Rightarrow V \propto \dfrac{1}{p} = p^{-1}$

Not quite right because this assumes that T is constant.

It is not true in sound wave.

(2) The compression is actually adiabatic meaning that there is almost NO HEAT flowing IN and OUT of the volume.

The time scale of the heat flow is larger than the time scale of oscillation

$\Rightarrow \quad pV^{\gamma} = \text{const}$

$pV^{\gamma} = C$

Consider : small vibration

$\psi_p \ll P_0$ $\qquad$ ψ_p: change in pressure w.r. P_0

$\Delta V \ll V_0$

Before : $\quad P_0 V_0^{\gamma} = C \quad$ — ①

After : $\quad (P_0 + \Delta P)(V_0 + \Delta V)^{\gamma} = C \quad$ — ②

Georgi (7.3)

② ⇒ $$C = (P_0 + \overset{\Delta P}{\overset{\downarrow}{\psi_p}})\, V_0^{\gamma} \left(1 + \frac{\Delta V}{V_0}\right)^{\gamma}$$

$$\approx (P_0 + \psi_p)\, V_0^{\gamma} \left(1 + \frac{\gamma \Delta V}{V_0}\right)$$

$$\approx \underset{\overset{\parallel}{C}}{P_0 V_0^{\gamma}} + \gamma \Delta V\, V_0^{\gamma-1} P_0 + \psi_p V_0^{\gamma} + \underbrace{\gamma \Delta V \psi_p}_{} V_0^{\gamma-1}$$

ignore $\Delta V \psi_p$ term (small)

② ⇒ $$\psi_p = \frac{-\gamma P_0}{V_0} \Delta V$$

Plugin the expression we got before for

② ⇒ $$\psi_p = \frac{-\gamma P_0\, A \Delta x}{V_0} \frac{\partial \psi}{\partial x}$$ ($V_0 = A \Delta x$)

② ⇒ $$\boxed{\psi_p = -\gamma P_0 \frac{\partial \psi}{\partial x}}$$ —— ③

Now we know how to relate the pressure change ψ_p and the displacement ψ

$\psi(x,t)$: displacement of the air with respect to the equilibrium position x

$\psi_p(x,t)$: "displacement" or change in pressure with respect to the room pressure P_0

Force acting on this volume of air :

$$F_{total} = \Delta P \cdot A$$

$$= -A \frac{\partial \psi_p}{\partial x} \Delta X$$ (from page 1)

Mass : $\Delta m = \rho \cdot A \cdot \Delta X$

* Newton's Law $F = ma$

$$\rho A \Delta x \ddot{\psi} = -A \Delta X \frac{\partial \psi_p}{\partial X}$$

$$\rho \ddot{\psi} = \frac{\partial \psi_p}{\partial X}$$

$$= \gamma P_0 \frac{\partial^2 \psi}{\partial X^2}$$ (from ③)

$$\Rightarrow \ddot{\psi}(x,t) = \frac{\gamma P_0}{\rho} \frac{\partial^2 \psi(x,t)}{\partial X^2}$$

Wave Equation !!!

$$v_p = \sqrt{\frac{\gamma P_0}{\rho}}$$

Adiabatic index γ

slide 11

* First law of thermodynamics

$$dU + \delta W = \delta Q$$

U : internal energy
W : work done by the sys.
Q : heat supplied to the sys.

* Adiabatic process:

$$dU + \delta W = 0$$

$$\delta W = P\,dV$$

Equipartition of Energy

$$U = \alpha\, nRT \qquad \alpha : \frac{\text{degrees of freedom}}{2}$$

$$= \alpha PV$$

$$dU = \alpha\,(dP\,V + P\,dV) = -\delta W = -P\,dV$$

$$(\alpha+1)\,P\,dV = -\alpha\, V\,dP$$

$$\frac{dP}{P} = -\left(\frac{\alpha+1}{\alpha}\right)\frac{dV}{V} = -\gamma\,\frac{dV}{V} \qquad \gamma \equiv \frac{\alpha+1}{\alpha}$$

$$\Rightarrow \quad PV^{\gamma} = \text{const}$$

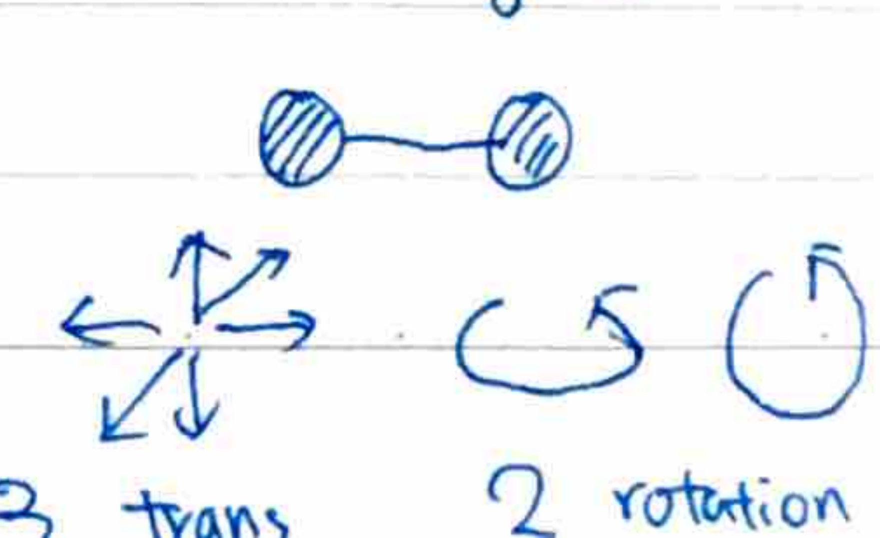
$\alpha = 5/2$
$\gamma = 7/5$

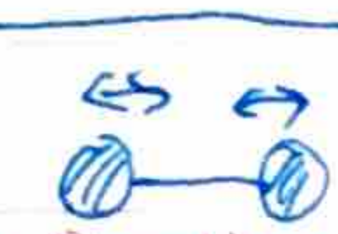
monoatomic gas

$\alpha = \frac{3}{2} \Rightarrow \gamma = 5/3$

3 translational degrees of freedom

diatomic gas





Not excited until high T

γ for diatomic gas is $7/5$

Air at sea level: $P_0 \approx 10^5 \ kg/ms^2$

Air density : $\rho = 1.2 \ kg/m^3$

$\Rightarrow$ Speed of sound: $v_p = 342 \ m/s$

Experiment : $v_p = 343 \ m/s$ at 70 F

Very nice agreement !!

What have we learned ?

(A) The speed of sound increases

if we use monatomic gas
to replace diatomic gas (Air)

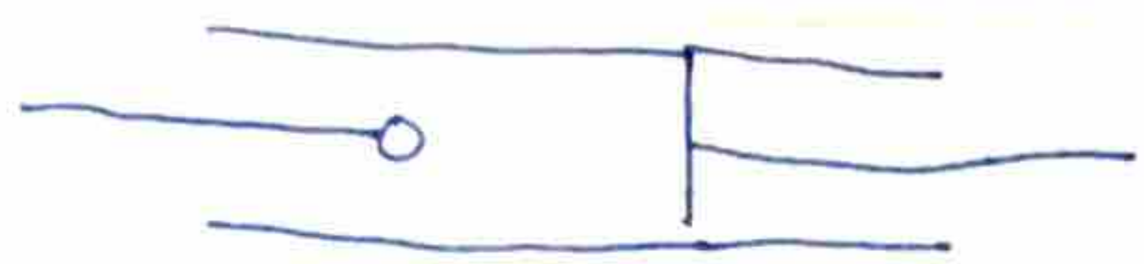
$\gamma \rightarrow$ increases

If wave length is fixed

$\Rightarrow$ higher frequency

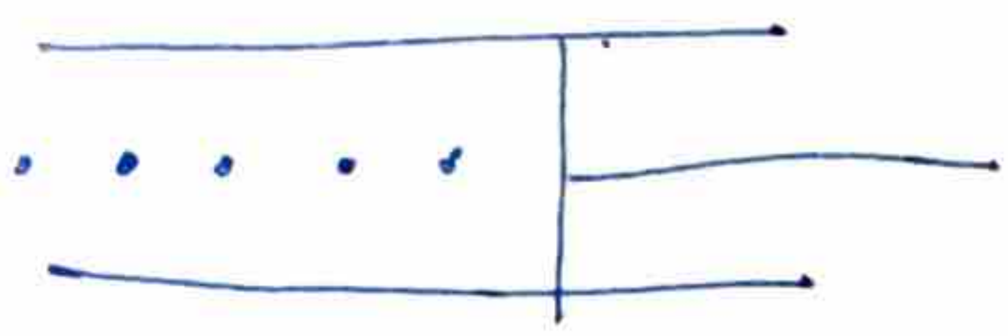
DEMO

DEMO



microphone

Estimation of what to expect.



$$\lambda = \frac{v}{f} \qquad \frac{360\ m/s}{1000} = 0.34\ m$$

Search for 4 modes $\Rightarrow$ 4λ

$$\Rightarrow \quad v = f \cdot \lambda$$

$$64_{cm} - 30_{cm} = 34\ cm$$

$$f = 1\ khz$$

$$\Rightarrow \quad v = 34_{cm} \cdot 1\ kHz$$

$$= 34k\ cm/s$$

$$= 340\ m/s$$

VERY CLOSE to the calculated result in page 6 !!! (342 m/s)

Rule out the prediction from Newton !

DEMO

(B) The fact that they are described by wave eq:

(1) $\omega = v_p K$

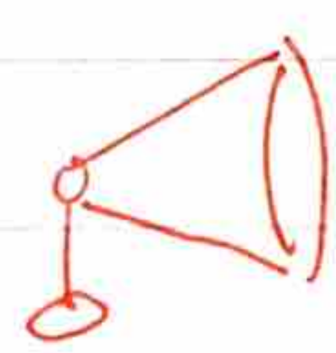
(2) Normal modes:

$$\psi(x) = \sum_{m=1}^{\infty} A_m \sin(K_m x + \alpha_m) \sin(\omega_m t + \beta_m)$$

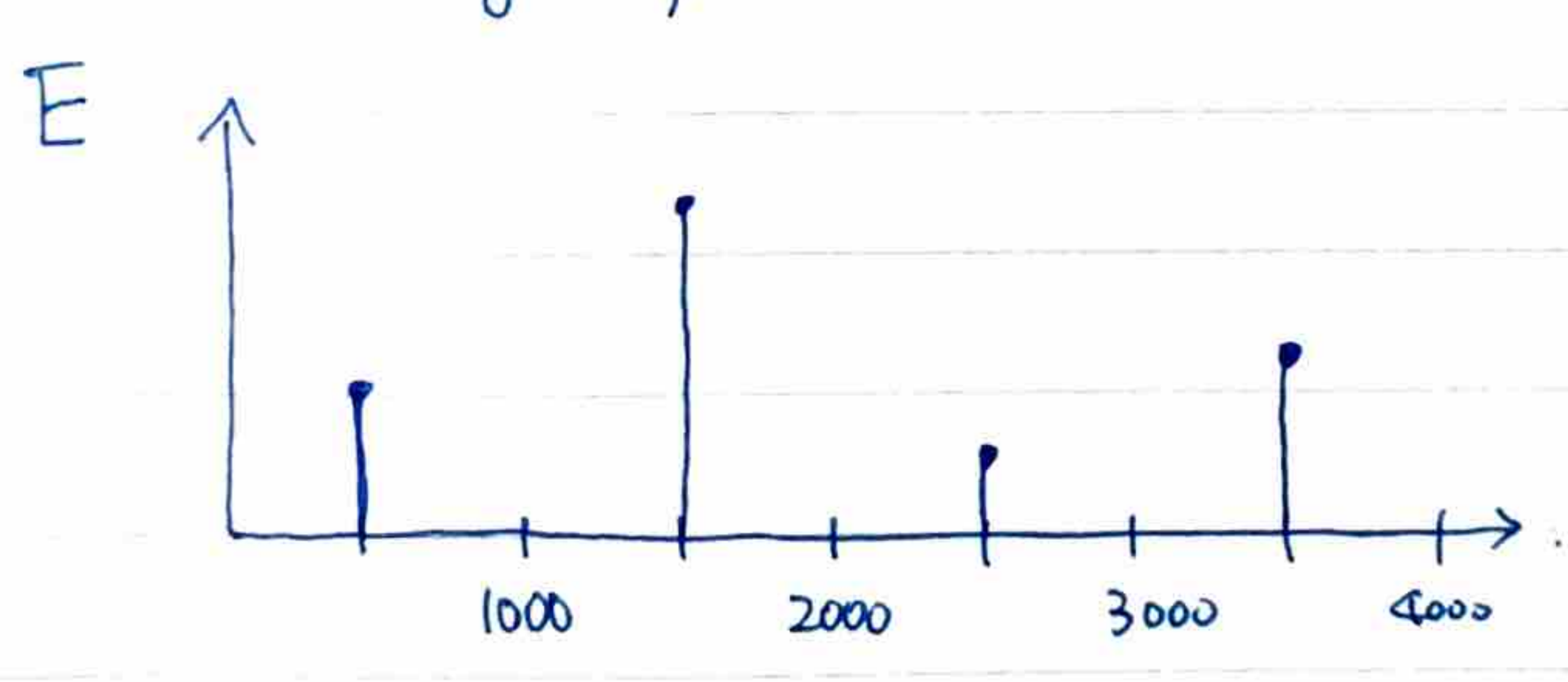
(3) K_m, α_m: determined by boundary conditions

(4) A_m, β_m: determined by initial conditions.

Break?

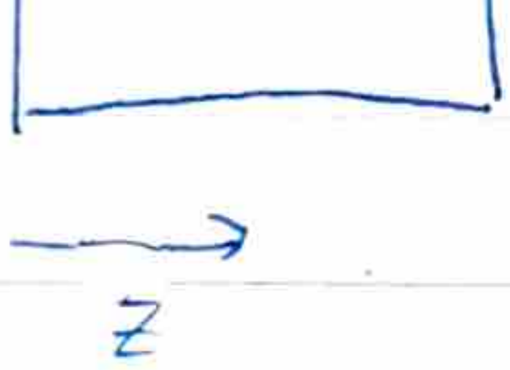
Example:   Organ pipe

Audio analyzer recorded the following energy vs frequency:

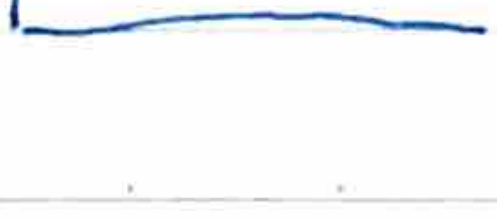


(i) Which configuration gave rise this power spectrum?

(A) L



(B) L



(C) L

Boundary condition for a closed end:

$\psi = 0$ $\because$ air can go nowhere :)

Boundary condition for an open end

$\frac{\partial \psi}{\partial z} = 0$ $\because$ Pressure = room pressure.

(A) ① $\psi(0) = 0$, ② $\psi(L) = 0$ $\psi_m = A_m \sin(k_m x + \alpha_m) \sin(\omega_m t + \beta_m)$

① $\Rightarrow$ $\sin(\alpha_m) = 0$

$\Rightarrow$ $\alpha_m = 0$

② $\Rightarrow$ $\sin(k_m L) = 0$ $\Rightarrow$ $k_m = \frac{m\pi}{L}$

$\Rightarrow$ $\omega_m = \frac{m\pi v}{L}$

(B) ① $\psi(0) = 0$ ② $\frac{\partial \psi(L)}{\partial z} = 0$ again $\alpha_m = 0$

$\Rightarrow$ $\cos(k_m x) = 0$ $\Rightarrow$ $k_m = \frac{(m-\frac{1}{2})\pi}{L}$

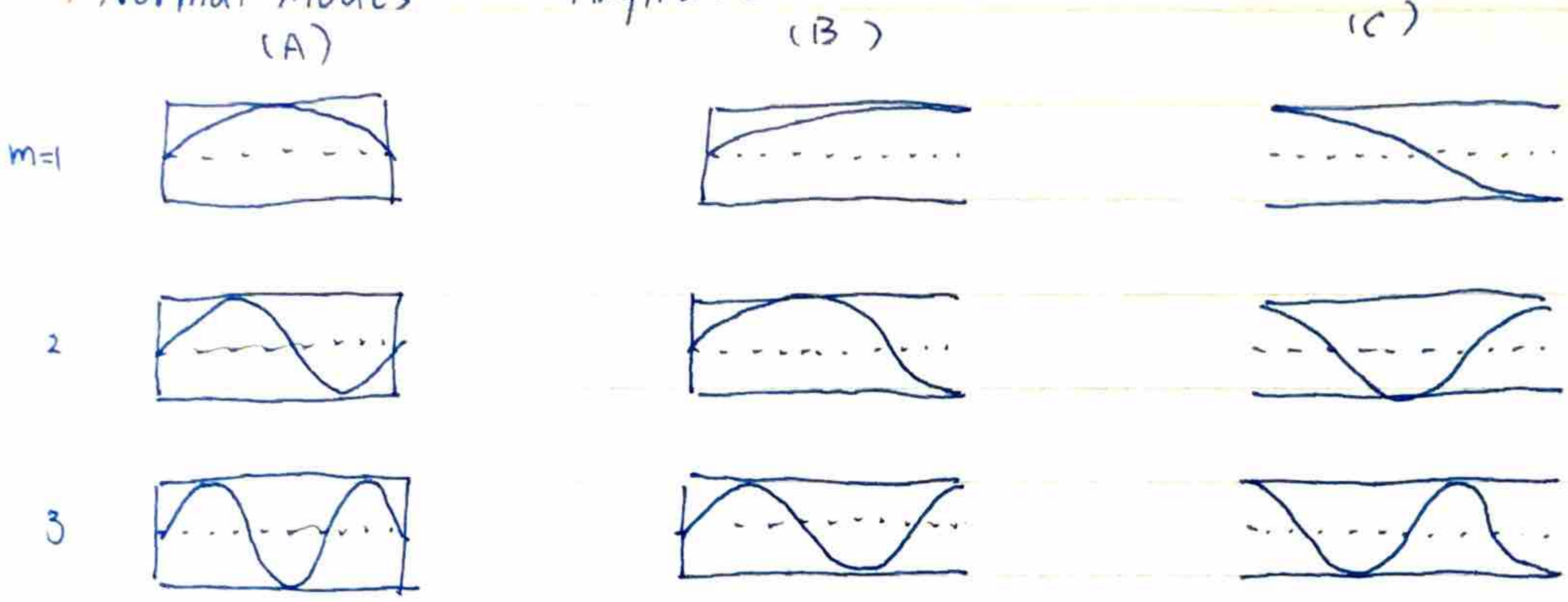
$\omega_m = \frac{(m-\frac{1}{2})\pi v}{L}$ $\Rightarrow$ Match the data!

(C) ① $\frac{\partial \psi(0)}{\partial z} = 0$ ② $\frac{\partial \psi(L)}{\partial z} = 0$

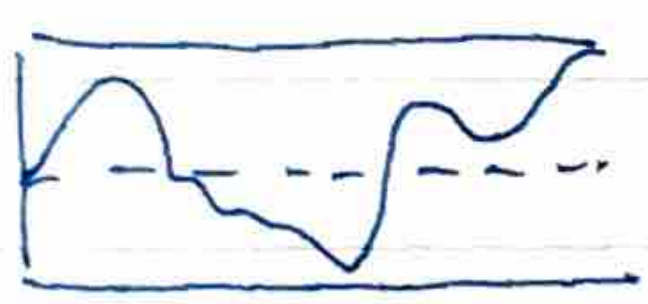
① $\Rightarrow$ $\cos(\alpha_m) = 0$ $\Rightarrow$ $\alpha_m = \frac{\pi}{2}$

② $\Rightarrow$ $\sin(k_m L + \frac{\pi}{2}) = 0$ $\Rightarrow k_m = \frac{m\pi}{L}$ $\omega_m = \frac{m\pi v}{L}$

(ii) Normal modes : Amplitude:

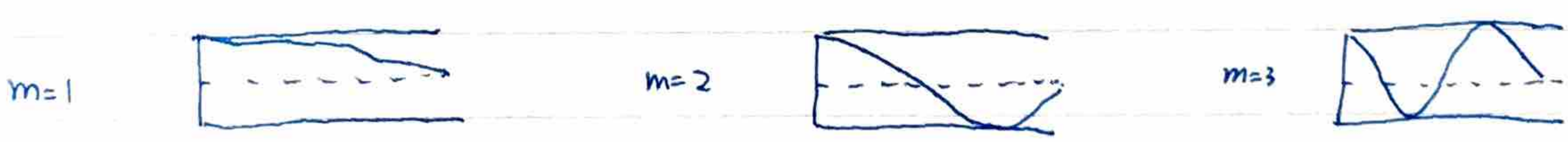


(iii) How long does it takes for this amplitude pattern



to reappear ? $\Rightarrow \frac{2\pi}{\omega_1}$

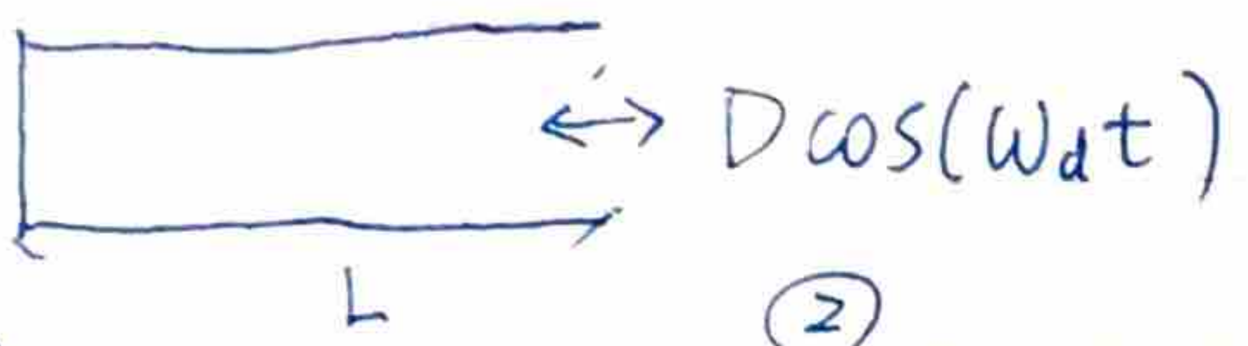
(iv) What about pressure ?
in each normal mode ?



$$\psi_p = \gamma P_0 \frac{\partial \psi}{\partial x}$$

(Opposite Pattern v.s. Amplitude.)

(V) Drive the organ



① $\psi(0)=0$, ② $\psi(L) = D\cos(\omega_d t)$

$$k_d = \frac{\omega_d}{v}$$ (decided by the dispersion relation $\omega = v \cdot k$)

$$\psi(x) = A_d \sin(k_d x + \alpha)\cos(\omega_d t)$$

① $\psi(0)=0 \Rightarrow \alpha = 0$

② $\psi(L) = D\cos(\omega_d t)$

$$\Rightarrow A_d \sin(k_d L) = D$$

$$A_d = \frac{D}{\sin(k_d L)}$$

$$\Rightarrow \psi(x) = \frac{D}{\sin(k_d L)} \sin(k_d x)\cos(\omega_d t)$$

when $k_d = \frac{(m-\frac{1}{2})\pi}{L}$ $\Rightarrow$ resonance!

(Huge amplitude)!