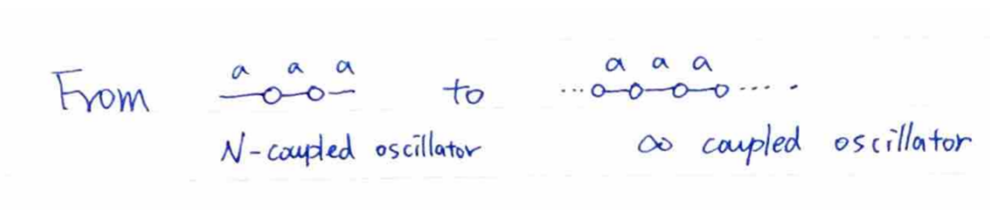
8.03 Lecture 8

This is what we have done:

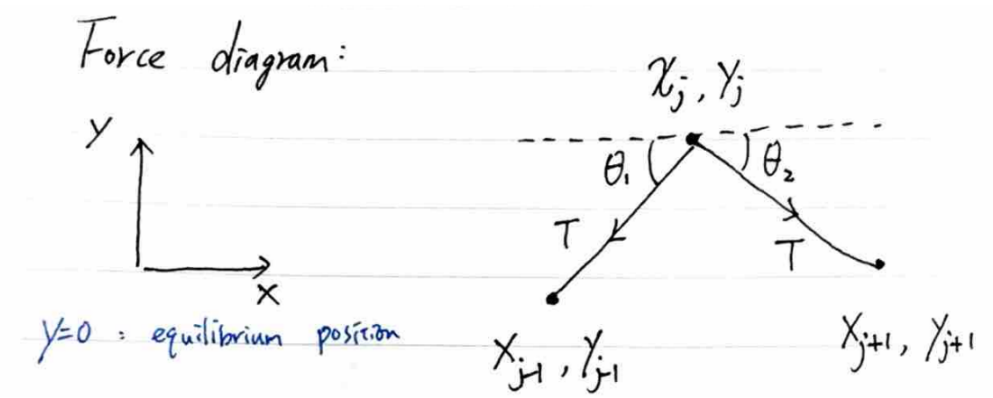


And we go from N coupled equations of motion to and infinite number of coupled equations of motion. Idea: we can make use of the **"Space Translation Invariance"**. This symmetry can be translated into mathematics: $A' = SA$ such that $A'_j = A_{j+1}$
If A is an eigenvector of S:

$$\begin{aligned} SA &= \beta A \\ A'_j &= \beta A_j = A_{j+1} \\ A_j &= \beta^j A_0 \propto \beta^j \end{aligned}$$

Consider $\beta = e^{ika}$ (Recall: we need $|\beta| = 1$ so that A_j does not go to ∞ when $j \to \infty$)
We get $A_j \propto e^{ijka}$
Let's consider this example:

A lot of point-like massive particles connected by a massless string, separated a distance a. These particles can only move up and down. We have constant tension T and small vibrations. Question: what will be the resulting motion of the system?
Force diagram:



Assume $y_j << a \quad \Rightarrow \quad (\theta_1, \theta_2) << 1$
Horizontal direction: $m\ddot{x}_j = -T\cos\theta_1 + T\cos\theta_2$
Vertical direction: $m\ddot{y}_j = -T\sin\theta_1 - T\sin\theta_2$
Since θ_1 and θ_2 are small $\Rightarrow \cos\theta \approx 1$ and $\sin\theta \approx \theta$

$$
\begin{aligned}
m\ddot{x}_j &= -T + T = 0 \quad \text{(No motion in the horizontal direction)} \\
m\ddot{y}_j &= -T(\sin\theta_1 + \sin\theta_2) \\
&\approx -T(\frac{y_j - y_{j-1}}{a} + \frac{y_j - y_{j+1}}{a}) \\
m\ddot{y}_j &= \frac{T}{a}(y_{j-1} - 2y_j + y_{j+1})
\end{aligned}
$$

Normal modes: $y_j = \mathrm{Re}[A_j e^{i(\omega t + \phi)}]$

From the S matrix, the eigenvectors are $A = \begin{pmatrix} \vdots \\ A_j \\ A_{J+1} \\ \vdots \end{pmatrix}$

$$A_j \propto \beta^j = e^{ijka}$$

Reminder: a is the distance between particles in the $\hat{x}$ direction. To get $M^{-1}k$ matrix:

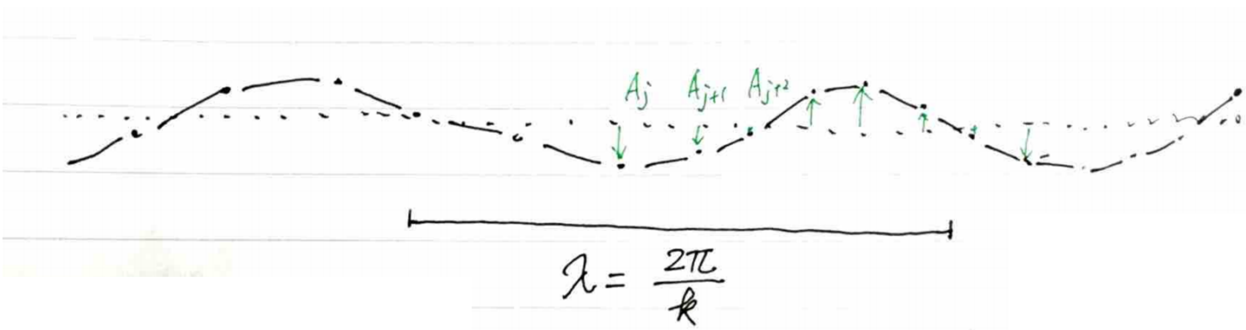
$$
M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \ddots \end{pmatrix} \quad k = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\frac{T}{a} & \frac{2T}{a} & -\frac{T}{a} & 0 & \cdots \\ \cdots & 0 & -\frac{T}{a} & \frac{2T}{a} & -\frac{T}{a} & \cdots \\ \cdots & 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}
$$

$$
M^{-1}k = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\frac{T}{ma} & \frac{2T}{ma} & -\frac{T}{ma} & 0 & \cdots \\ \cdots & 0 & -\frac{T}{ma} & \frac{2T}{ma} & -\frac{T}{ma} & \cdots \\ \cdots & 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}
$$

To get ω, since $M^{-1}k$ and S share the same eigenvectors:
Calculate $M^{-1}kA = \omega^2 A$. The jth term:

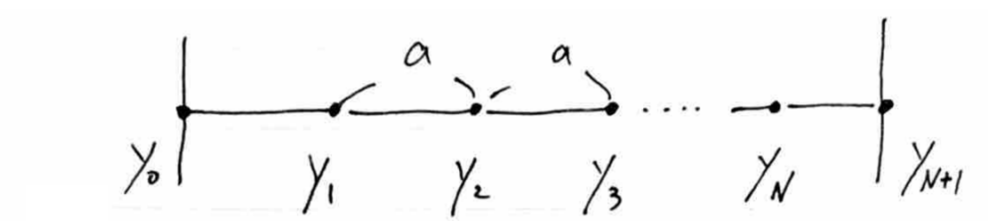
$$\begin{aligned}
\omega^2 A_j &= \frac{T}{ma}(-A_{j-1} + 2A_j - A_{j+1}) \\
\omega^2 A_j &= \frac{T}{ma} A_j(-e^{-ika} + 2 + e^{ika} \\
\omega^2 &= \frac{T}{ma}(2 - 2\cos ka) \\
&= 2\omega_0^2(1 - \cos ka) \\
\omega^2 &= 4\omega_0^2 \sin^2\left(\frac{ka}{2}\right)
\end{aligned}$$

Where we have defined $\omega_0^2 \equiv T/ma$. This is almost the same result that we got from last lecture! $\omega = \omega(k)$, or ω is a function of k. This is known as a "dispersion relation." When k (the wavenumber $k = 2\pi/\lambda$) is given then ω (the angular frequency) is determined.
Our normal modes are standing waves:



Oscillating at frequency ω as determined by k

This system is infinitely long. All possible k values (wavelengths) are allowed. Each k value corresponds to a different normal mode with angular frequency given by $\omega(k)$.
Now we still try to solve a finite system using the solution for the infinite system. Consider the following boundary condition:
(1) Fixed end:



Boundary conditions: $y_0 = 0 \quad y_{N+1} = 0$

What are the normal modes that satisfy the boundary conditions? There are two values of k which give the same ω

$$\omega(k) = \omega(-k)$$

Therefore: linear combinations of e^{ijka} and e^{-ijka} are also normal modes. Guess:

$$y_j = \text{Re}\left[e^{i(\omega t+\phi)}(\alpha e^{ijka} + \beta e^{-ijka})\right]$$

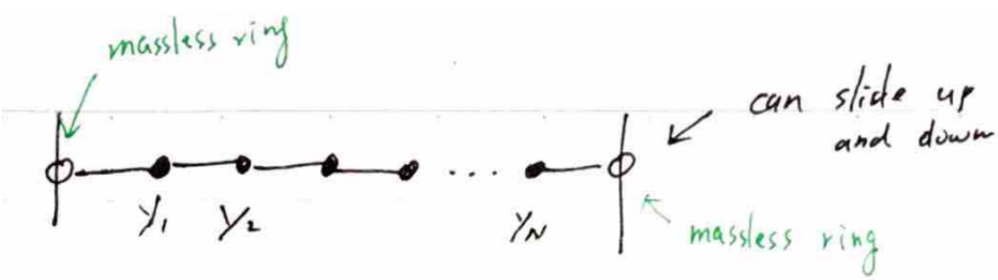
Where α and β are constants. Use the boundary conditions:

$$y_0 = 0 \;\Rightarrow\; \alpha + \beta = 0 \;\Rightarrow\; \alpha = -\beta$$

$$y_{N+1} = 0 \;\Rightarrow\; \alpha\left(e^{i(N+1)ka} + e^{-i(N+1)ka}\right) = 0$$

$$2i\sin(N+1)ka = 0 \;\Rightarrow\; ka = \frac{n\pi}{N+1}$$

Where n is a positive integer less than N
(More examples:)
(2) Open end:



Boundary conditions: $y_1 = y_0 \quad y_{N+1} = y_N$

From first boundary condition we get:

$$\alpha + \beta = \alpha e^{ika} + \beta e^{-ika}$$
$$\alpha(1 - e^{ika}) = \beta(e^{-ika} - a)$$

Second boundary condition:

$$\alpha e^{iNka} + \beta e^{-iNka} = \alpha e^{i(N+1)ka} + \beta e^{-i(N+1)ka}$$
$$\alpha e^{iNka}(1 - e^{ika}) = \beta e^{-iNka}(e^{-ika} - 1)$$

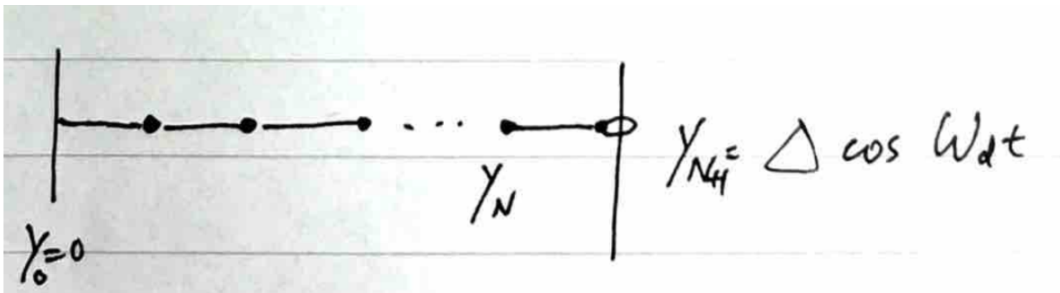
Dividing the first condition by the second condition:

$$e^{iNka} = e^{-iNka}$$
$$\Rightarrow e^{2iNka} = 1$$

$$\Rightarrow ka = \frac{2n\pi}{2N} = \frac{n\pi}{N}$$

$$
\begin{aligned}
\beta &= \alpha e^{ika} \\
y_j &= \alpha(e^{ijka} + e^{-i(j-1)ka}) \\
&= \alpha e^{-ika/2}(e^{i(j-1/2)ka} + e^{-i(j-1/2)ka}) \\
&\propto \cos(ka(j-1/2))
\end{aligned}
$$

(3)



Boundary conditions: $y_0 = 0 \quad y_{N+1} = \Delta \cos \omega_d t$

Need to find the "particular solution"
y_j must be oscillating at a frequency ω_d
What is the corresponding k_d which gives ω_d? Use $\omega(k)$:

$$\omega_d^2 = 2\omega_0^2(1 - \cos k_d a)$$

Solve to get $k_d a = \cos^{-1}\left(1 - \frac{\omega_d^2}{2\omega_0^2}\right)$
Guess:

$$y_j = \text{Re}\left[e^{i\omega_d t}(\alpha e^{ijk_d a} + \beta e^{-ijk_d a})\right]$$

Use the boundary condition at $j = 0$:

$$y_0 = 0 \;\Rightarrow\; \alpha + \beta = 0 \;\Rightarrow\; \beta = -\alpha$$

$$y_j = \text{Re}\left[2ie^{i\omega_d t} A \sin jk_d a\right]$$

Use the boundary condition at $j = N+1$:

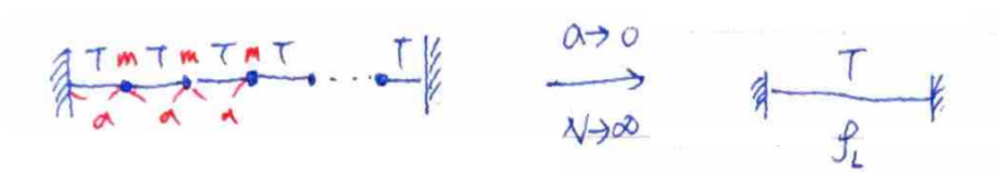
$$
\begin{aligned}
y_{N+1} &= Z \cos \omega_d t = \text{Re}[\Delta e^{i\omega_d t}] \\
\Rightarrow \;\; \Delta &= 2iA \sin(N+1)k_d a \\
A &= \frac{\Delta}{2i \sin(N+1)k_d a} \\
\Rightarrow \;\; y_j &= \text{Re}\left[\frac{\Delta \sin jk_d a}{\sin(N+1)k_d a} e^{i\omega_d t}\right] \\
&= \frac{\Delta \sin jk_d a}{\sin(N+1)k_d a} \cos \omega_d t
\end{aligned}
$$

Which explodes when $k_d a = \frac{n\pi}{N+1}$!! (When the driving force matches the normal mode frequency)

Summary:

1. Symmetry + does not explode at the edge of the universe choose $\Rightarrow \beta = e^{ika}$
2. Equation of motion can be derived from physical laws
3. Dispersion relation $\omega(k)$ can be derived from items 1 and 2
4. The allowed k value is determined by boundary conditions. The full solution is a linear combination of normal modes
5. Use initial conditions to determine unknowns

Now make it continuous!!!



jth term of

$$M^{-1}kA \;\Rightarrow\; \omega^2 A_j = \frac{T}{ma}(-A_{j-1} + 2A_j - A_{j+1})$$

In the continuous limit this equation transforms into:

$$M^{-1}kA \;\Rightarrow\; \omega^2 A(x) = \frac{T}{ma}(-A(x-a) + 2A(x) - A(x+a))$$

If we Taylor Expand:

$$A(x-a) = A(x) - aA'(x) + \frac{1}{2}a^2 A''(x) + \cdots$$
$$A(x+a) = A(x) + aA'(x) + \frac{1}{2}a^2 A''(x) + \cdots$$
$$\Rightarrow \quad -A(x-a) + 2A(x) - A(x+a) = -\frac{\partial^2 A(x)}{\partial x^2}a^2 + \cdots$$
$$M^{-1}kA(x) = -\frac{T}{ma}\frac{\partial^2 A(x)}{\partial x^2}a^2 + \cdots$$

In the $a <<$ wavelength we can ignore the a^3 and higher order terms. We define $\rho_L \equiv \frac{m}{a}$ and $M^{-1}k$ becomes an "operator":

$$M^{-1}k \to -\frac{T}{\rho_L}\frac{\partial^2}{\partial x^2}$$
$$\frac{\partial\psi(x,t)}{\partial t^2} = \frac{T}{\rho_L}\frac{\partial\psi(x,t)}{\partial x^2}$$

In the last equation we plug in the normal mode $e^{ikx}e^{i\omega t}$
Dispersion relation:

$$\omega^2 = \frac{T}{\rho_L}k^2$$

$$\frac{\omega}{k} = v_p = \sqrt{\frac{T}{\rho_L}}$$

Where v_p is the phase velocity, ω is the angular frequency and k is the wave number.