8.03 Lecture 13

Reminder: Maxwell's equation in vacuum

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

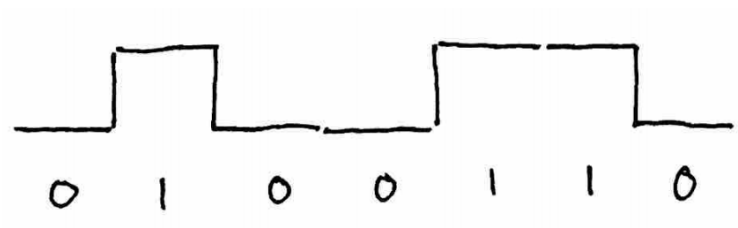
Where $c \equiv 1/\sqrt{\mu_0 \epsilon_0}$
Resulting wave equations:

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\vec{\nabla}^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

We discussed plane harmonic wave solution. And you will show that in general a progressing wave solution:

$$\vec{E} = E_0 \hat{y} f(z - vt)$$

and the corresponding $\vec{B}$ field also satisfies Maxwell's equations.
How do we transmit "information"? A simple harmonic wave would not be useful. We must use "pulses," chunks of localized energy in time. For instance:

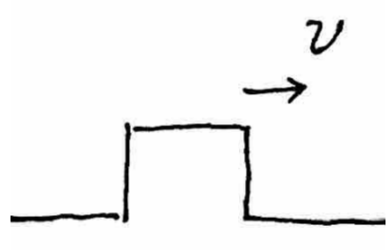


We have learned:
$f(x - vt)$ or $f(kx - \omega t)$ is a traveling wave moving in the $+\hat{x}$ direction and its shape is kept unchanged if and only if we are working in a non dispersive medium, i.e. $\omega/k = v$
Consider an ideal string:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

Where

$$\frac{\omega}{k} = v = \sqrt{\frac{T}{\rho_L}}$$



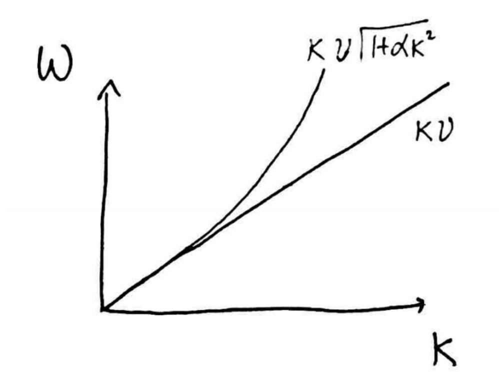
If we create a square pulse, the square pulse will move at constant speed v. The shape of the square pulse does not change! We call this string a non-dispersive medium and the "dispersion relation" is $\omega = vk$. Note: the string tension is responsible for the restoring force.
However, if we consider the stiffness of the string, (for example, a piano string): If we bend a piano string, even when there is no tension, the string tends to restore to its original shape. To model "stiffness":

$$\frac{\partial^2\psi}{\partial t^2} = v^2\left[\frac{\partial^2\psi}{\partial x^2} - \alpha\frac{\partial^4\psi}{\partial x^4}\right]$$

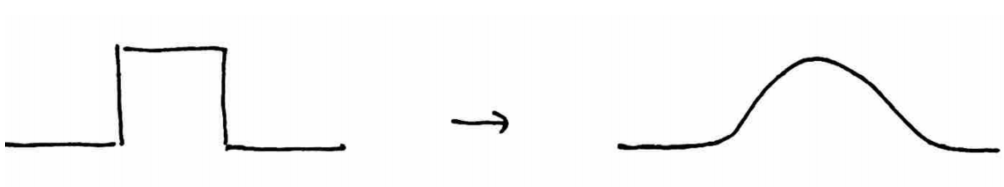
The dispersion relation becomes (where we use $A\cos(kx-\omega t)$ as a test function):

$$\omega^2 = v^2(k^2+\alpha k^4)$$
$$\Rightarrow \quad \frac{\omega}{k} = v\sqrt{1+\alpha k^2}$$

Not a constant versus k anymore!!



Where $k = 2\pi/\lambda$. Large $k \Rightarrow$ short $\lambda \Rightarrow$ a lot of dispersion and a higher speed v
As a consequence, components with different k will be moving at different speeds $v_p = \omega(k)/k$ and we get a dispersion, or the wave loses shape:



Dispersion is a variation of wave speed with wave length. Example: addition of two progressing waves:

$$\psi_1(x,t) = A\sin(k_1x-\omega_1t) \qquad v_1 = \frac{\omega_1}{k_1}$$

$$\psi_2(x,t) = A\sin(k_2x-\omega_2t) \qquad v_2 = \frac{\omega_2}{k_2}$$

If we add $\psi_1 + \psi_2$ and using the trig identity

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A+B)$$

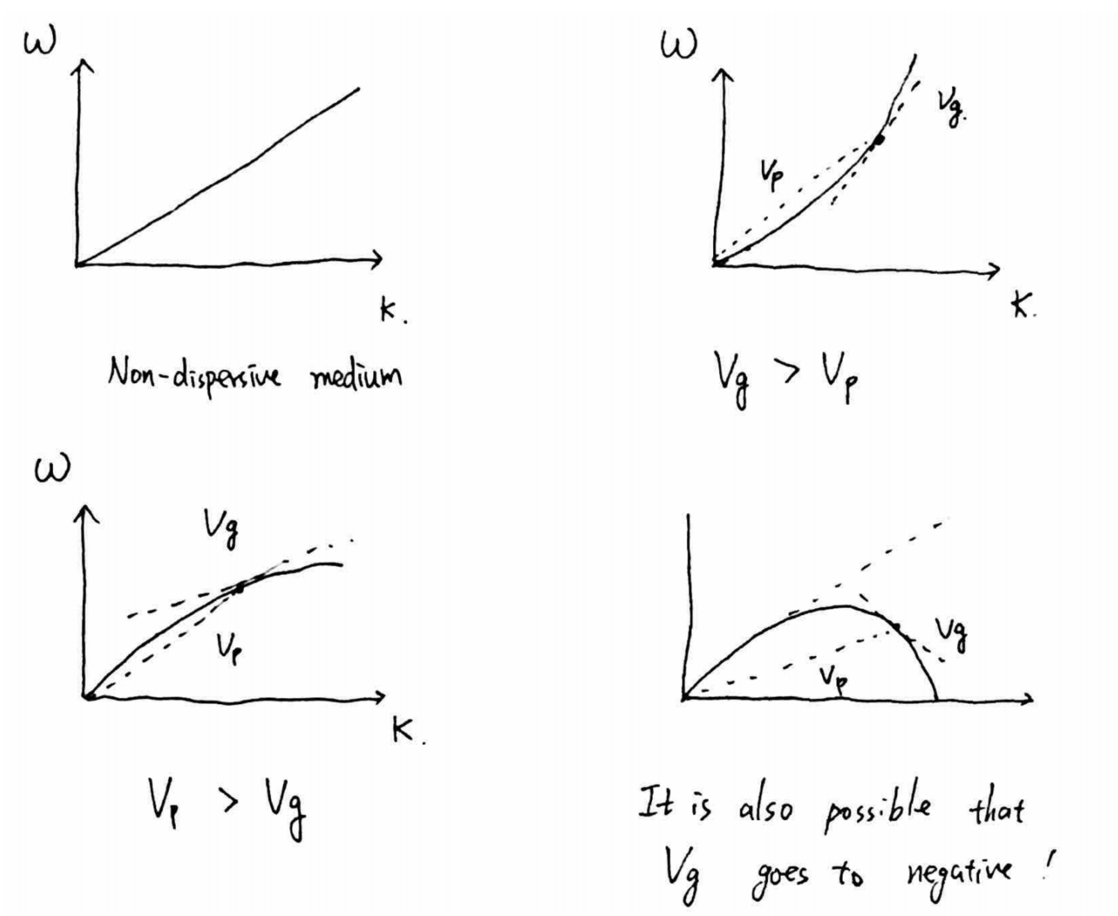
we get

$$\psi_1 + \psi_2 = 2A \sin\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right) \cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)$$

Assuming $k_1 \approx k_2 \approx k$ and $\omega_1 \approx \omega_2 \approx \omega$ we have “amplitude modulation:

Where the phase and group velocity is

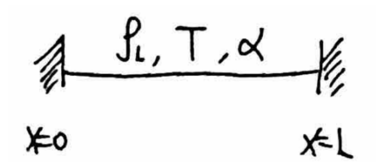
$$v_p = \frac{\omega}{k} \qquad v_g = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)} \approx \frac{d\omega}{dk}$$



Bounded system:

$$\psi(x,t) = \sum_m A_m \sin(k_m x + \alpha_m)\sin(\omega_m t + \beta_m)$$

Where $\omega_m = \omega(k_m)$, then evolve as a function of time!
Now consider the boundary conditions of this system:



$$\psi(0,t) = 0 \qquad \& \qquad \psi(L,t) = 0$$

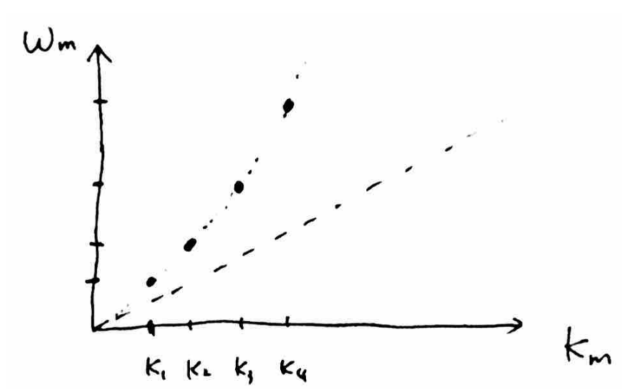
This is similar to something we have solved before, and we got:

$$k_m = \frac{m\pi}{L} \qquad , \qquad \alpha_m = o$$

Identical to the ideal string case ($\alpha = o$) We learned that:

1. The boundary condition "set" the k_m! Does not depend on the dispersion relation $\omega(k)$
2. The individual normal modes are oscillating at $\omega_m = \omega(k_m)$ as calculated by the dispersion relation: This does depend on the dispersion relation!

If we plot the dispersion relation:



But in general ω_m is not equally spaced.
Full solution:

$$\psi(x,t) = \sum_m A_m \sin(k_m x + \alpha_m)\sin(\omega_m t + \beta_m)$$
$$= \sum_m \psi_m$$

Example: $\psi(x,t) = \psi_1 + \psi_2$

In a non-dispersive medium: the system goes back to the original shape after $2\pi/\omega_1$
In a dispersive medium $\omega_2 \neq \omega_1$. We need to wait longer until the reaches the least common multiple of $2\pi/\omega_1$ and $2\pi/\omega_2$