8.03 Fall 2016 Practice Exam 2 Solution

Problem 1

a.

$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \frac{\partial^2 \rho}{\partial x^2} - \omega_p^2 \rho. \tag{1}$$

Plug in $\rho(x,t) = a\sin(kx - \omega t)$, we get

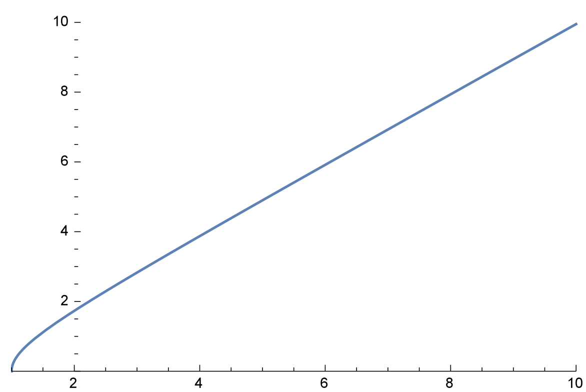
$$-\omega^2 = -c^2 k^2 - \omega_p^2 \tag{2}$$

Hence

$$\omega = \sqrt{c^2 k^2 + \omega_p^2} \tag{3}$$

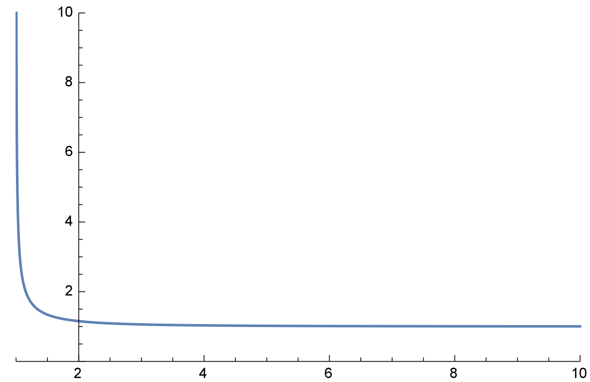
b.

$$k(\omega) = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}. \tag{4}$$



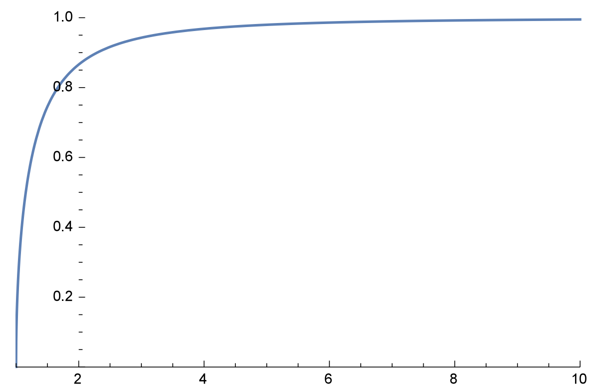
The phase velocity

$$v = \frac{\omega}{k} = c\frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \tag{5}$$



The group velocity

$$v = (\frac{dk}{\omega})^{-1} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{6}$$



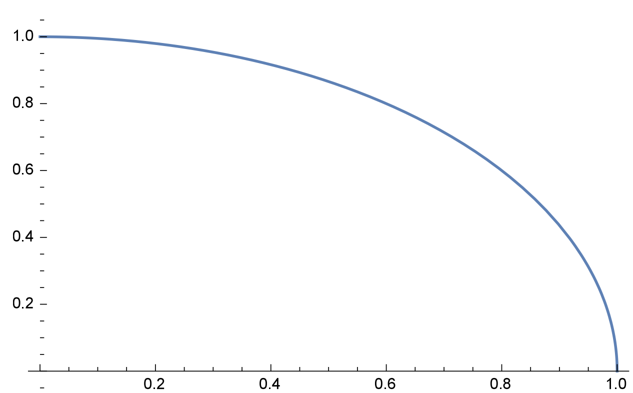
c.

Plug $\rho(x,t) = a\cos(\omega t)e^{-\kappa x}$ in to the wave equation, we get

$$-\omega^2 = c^2\kappa^2 - \omega_p^2 \tag{7}$$

Hence

$$\kappa = \frac{1}{c}\sqrt{\omega_p^2 - \omega^2} \tag{8}$$



Problem 2

a.

The tension force from the string in the vertical direction

$$T_y = T\frac{dy}{dx}|_{x=0} \tag{9}$$

balances the external force $F(t)$. Hence the boundary condition is

$$F_0 \sin(\omega t) + T\frac{dy}{dx}|_{x=0} = 0 \tag{10}$$

b.

The wave has to satisfy the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \tag{11}$$

where $v^2 = \frac{T}{\mu}$. The general complex solution with a single freqency is

$$y = f\cos(\omega'(t - \frac{x}{v}) + \phi_1) + g\cos(\omega'(t + \frac{x}{v}) + \phi_2) \tag{12}$$

For the steady-state solution to satisfy the boundary condition for arbitrary t, this ω' has to equal to ω. Hence the frequency of steady-state solution has angular frequency ω and wavelength

$$\lambda = \frac{2\pi v}{\omega}. \tag{13}$$

c.

We don't need to keep the reflected pulse term, hence just plug

$$y = f\cos(\omega(t - \frac{x}{v}) + \phi_1) \tag{14}$$

into the boundary condition

$$F_0 \sin(\omega t) + T\frac{\partial y}{\partial x} = 0, \tag{15}$$

we have

$$F_0 \sin(\omega t) + \frac{Tf\omega}{v}\sin(\omega t + \phi_1) = 0. \tag{16}$$

We can take $\phi_1 = \pi$, $f = \frac{F_0 v}{T\omega}$, hence the result steady-state harmonic wave is in the form of

$$y = \frac{F_0 v}{T\omega}\cos(\omega(t - \frac{x}{v}) + \pi), \tag{17}$$

with amplitude

$$A = \frac{F_0 v}{T\omega}. \tag{18}$$

Problem 3

a.

The boundary condition is:

$$\frac{\partial y}{\partial x}|_{x=0} = 0, \tag{19}$$

$$y(L,t) = 0. \tag{20}$$

The normal modes are then in form of

$$y(x) = \cos(kx), \tag{21}$$

with condition

$$\cos(kL) = 0. \tag{22}$$

Hence the normal mode m is

$$y_m = \cos(k_m x)\ ,\ k_m = (m - \frac{1}{2})\frac{\pi}{L}\ (m = 1, 2, \dots) \tag{23}$$

The frequency of a normal mode is related to k by

$$\omega = kv = k\sqrt{\frac{T}{\mu}}. \tag{24}$$

Hence

$$\omega_m = (m - \frac{1}{2})\frac{\pi}{L}\sqrt{\frac{T}{\mu}}, \tag{25}$$

$$\tau_m = \frac{2\pi}{\omega_m} = \frac{2L}{m - \frac{1}{2}}\sqrt{\frac{\mu}{T}}. \tag{26}$$

b.

$$\begin{aligned} A_m &= \frac{2}{L}\int_0^L y(x,0)\cos\left((m - \frac{1}{2})\frac{\pi x}{L}\right)dx \\ &= \frac{2}{L}\int_{L/4-D/2}^{L/4+D/2} H\cos\left((m - \frac{1}{2})\frac{\pi x}{L}\right)dx \\ &= \frac{2H}{\pi(m - \frac{1}{2})}\left[\sin((m - \frac{1}{2})\pi(\frac{1}{4} + \frac{D}{2L})) - \sin((m - \frac{1}{2})\pi(\frac{1}{4} - \frac{D}{2L}))\right] \\ &= \frac{4H}{\pi(m - \frac{1}{2})}\cos\frac{(m - \frac{1}{2})\pi}{4}\sin\frac{(m - \frac{1}{2})\pi D}{2L} \end{aligned} \tag{27}$$

c.

$A_m = 0$ when

$$\cos\frac{(m - \frac{1}{2})\pi}{4} = 0 \tag{28}$$

or

$$\sin\frac{(m - \frac{1}{2})\pi D}{2L} = 0. \tag{29}$$

The first condition can never happen for integral m. This is because no normal modes has $y_m(L/4) = 0$.

The section condition may happen if there's some m satisfying

$$\frac{(m - \frac{1}{2})D}{2L} = p \tag{30}$$

is an integer.

d.

$$y(x,t) = \sum_{m=1}^{\infty} \frac{4H}{\pi(m-\frac{1}{2})} \cos\frac{(m-\frac{1}{2})\pi}{4} \sin\frac{(m-\frac{1}{2})\pi D}{2L} \cos\left((m-\frac{1}{2})\frac{\pi x}{L}\right) \cos((m-\frac{1}{2})\frac{v\pi}{L}t). \quad (31)$$

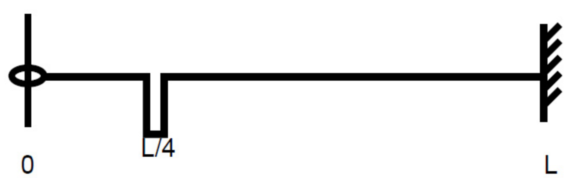
The time factors in each component are chosen to be $\cos(\omega_m t)$, so that the time derivative $\frac{\partial y}{\partial t}(x,0) = 0$.

e.

$$\tau_1 = \frac{4L}{v}\ ,\ t = \frac{\tau_1}{2} = \frac{2L}{v}. \quad (32)$$

The initial configuration of static string can be decomposed to two square pulses traveling in the opposite direction, each with velocity v, width D and height $\frac{H}{2}$. After time t, each of them have travelled back to the original position, hitting the wall once. Since hitting the wall will add a π phase shift, or turn the pulse upside down, while hitting the massless ring won't change the shape of the pulse, each of the two traveling pulses was turned upside down after time t.

Hence the shape of string at time $t = \frac{\tau_1}{2}$ is:



Problem 4

a.

The wave equation on the left:

$$\frac{\partial^2 y_L}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y_L}{\partial t^2}. \quad (33)$$

The wave equation on the right:

$$\frac{\partial^2 y_R}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y_R}{\partial t^2}. \quad (34)$$

The boundary condition at $x = 0$: the vertical component of string tension on the left and right should balance, hence

$$T\frac{\partial y_L}{\partial x}|_{x=0_-} = \frac{T}{2}\frac{\partial y_R}{\partial x}|_{x=0_+}. \quad (35)$$

We also have the continuity condition:

$$y_L|_{x=0_-} = y_R|_{x=0_+}. \quad (36)$$

b.

There are reflected waves at $x = 0$ and $x = L$. The reflected wave at $x = 0$ does not change the sign, since the impedance $\sqrt{T\mu}$ on the left is larger than the impedance on the right. The reflected wave at $x = L$ change the sign because it's a fixed boundary condition at $x = L$.

Hence these reflected waves have the opposite sign.

c.

We denote

$$v = v_1 = \sqrt{\frac{T}{\mu}}. \tag{37}$$

Hence the wave on the left of $x = 0$:

$$y_L = f_1(x/v - t) + f_2(x, t), \tag{38}$$

where the reflected wave

$$f_2(x, t) = r f_1(-x/v - t). \tag{39}$$

The wave on the right of $x = 0$:

$$y_R = g(x, t) + g_2(x, t). \tag{40}$$

Here $g(x, t)$ is the transmitted wave through $x = 0$:

$$g(x, t) = t f_1(x/v - t), \tag{41}$$

$g_2(x, t)$ is the reflected wave from $x = L$:

$$g_2(x, t) = r_2 f_1((2L - x)/v - t). \tag{42}$$

Then the boundary condition at $x = 0$ gives:

$$\frac{T}{v} f_1'(-t) - \frac{rT}{v} f_1'(-t) = \frac{tT}{2v} f_1'(-t) - \frac{r_2 T}{2v} f_1'(2L/v - t). \tag{43}$$

Since

$$f_1(t \geq 0) \equiv 0, \tag{44}$$

and we only consider times $t \leq 2L/v$, the last term in (43) vanishes, we have

$$1 - r = \frac{t}{2} \tag{45}$$

and

$$1 + r = t. \tag{46}$$

Hence

$$r = \frac{1}{3} \ , \ t = \frac{4}{3}. \tag{47}$$

At boundary $x = L$, we have

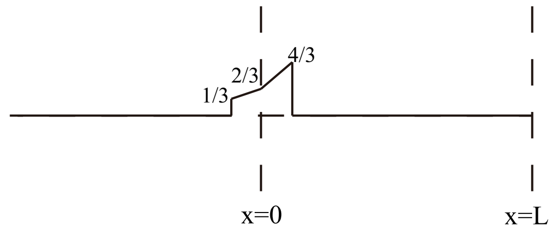
$$g(L, t) + g_2(L, t) = 0, \tag{48}$$

hence

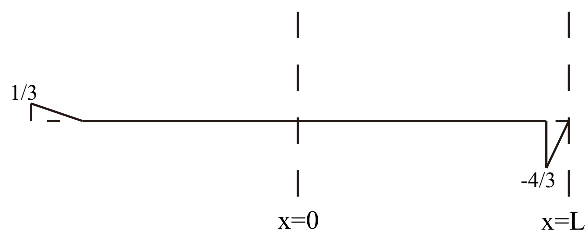
$$r_2 = -t = -\frac{4}{3}. \tag{49}$$

d.

At time $t = \frac{L}{v}$, the right half of the pulse has passed $x = 0$, also reflected to the left. The left half of the pulse is still propagating to the right. Denote the height of the right edge of the initial pulse by 1, the shape of string at $t = \frac{L}{v}$ is:



At the time $t = \frac{2L}{v}$, the reflected pulse f_2 has traveled back to $x = -L$, the transmitted pulse g has reached $x = L$. The right half of g is already reflected, while the left half is still propagating to the right. The shape of string at $t = \frac{2L}{v}$ is:



Problem 5

a.

The lowest mode is in the form

$$y_1 = \sin(k_1 x)\ ,\ \ k_1 = \frac{\pi}{L}. \tag{50}$$

The wavelength

$$\lambda_1 = 2L, \tag{51}$$

frequency

$$\omega_1 = vk_1 = k_1\sqrt{\frac{TL}{M}} = \frac{\pi}{L}\sqrt{\frac{TL}{M}}. \tag{52}$$

b.

The nth mode is in the form

$$y_n = \sin(k_n x)\ ,\ \ k_n = \frac{n\pi}{L}. \tag{53}$$

The wavelength

$$\lambda_n = \frac{2L}{n}, \tag{54}$$

frequency

$$\omega_n = vk_n = k_n\sqrt{\frac{TL}{M}} = \frac{n\pi}{L}\sqrt{\frac{TL}{M}}. \tag{55}$$

c.

The modes with $y_n(L/2) = 0$ will be zero, because they are antisymmetric at $x = L/2$:

$$y_n(L/2 - x) = -y_n(L/2 + x). \tag{56}$$

There shouldn't any contribution from these modes.

Hence the modes $n = 2m$ vanish.

d.

At $t = 0$, the string has zero speed.

The full function $y(x, t)$:

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cos\frac{n\pi vt}{L}. \tag{57}$$

After time $T = 2\frac{L}{v}$, all the time dependent factors $\cos\frac{n\pi vt}{L}$ returns to 1. This is the minimal time since $B_1 \neq 0$.

e.

The initial configuration can be decomposed into two traveling pulses to the opposite direction with velocity v, height $A/2$ and the same width as the initial configuration. When it hits the left or right boundary, it changes sign after reflected.

Hence at the time $t = \frac{L}{4v}$, the string configuration is:

which is a straight line.

f.

This time is $t = \frac{L}{4v}$ in the expression (57). Shift the zero point of time to this time, we have the expression for the 5th mode:

$$y_5(x,t) = B_5 \sin\frac{5\pi x}{L} \cos\frac{5\pi v(t + L/4v)}{L} = B_5 \sin\frac{5\pi x}{L} \cos\left(5\pi t\sqrt{\frac{T}{ML}} + \frac{5\pi}{4}\right). \tag{58}$$

g.

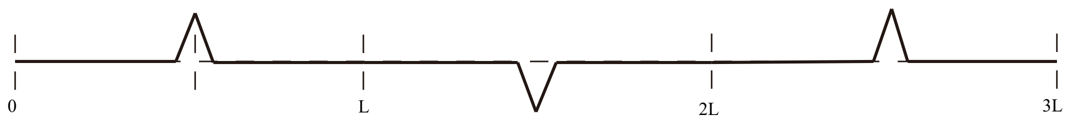
Since all the non-vanishing modes correspond to odd n, the function $f(x)$ in the range $L \sim 2L$ should be the opposite of function $f(x)$ in the range $0 \sim L$:

$$f(x+L) = \sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi(x+L)}{L} = -\sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi x}{L} = -f(x) \tag{59}$$

While the function $f(x)$ in the range $2L \sim 3L$ should be the same as function $f(x)$ in the range $0 \sim L$:

$$f(x+2L) = \sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi(x+2L)}{L} = \sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi x}{L} = -f(x). \tag{60}$$

Hence the shape of string:



h.

In this new configuration, the lowest mode is in the form

$$y_1 = \sin(k_1 x)\ ,\ k_1 = \frac{\pi}{2L}. \tag{61}$$

The wavelength

$$\lambda_1 = 4L, \tag{62}$$

frequency

$$\omega_1 = vk_1 = k_1\sqrt{\frac{TL}{M}} = \frac{\pi}{2L}\sqrt{\frac{TL}{M}}. \tag{63}$$

i.

The nth mode is in the form

$$y_n = \sin(k_n x)\ ,\ k_n = \frac{(2n-1)\pi}{2L}. \tag{64}$$

The wavelength

$$\lambda_n = \frac{4L}{(2n-1)}, \tag{65}$$

frequency

$$\omega_n = vk_n = k_n\sqrt{\frac{TL}{M}} = \frac{(2n-1)\pi}{2L}\sqrt{\frac{TL}{M}}. \tag{66}$$

j.

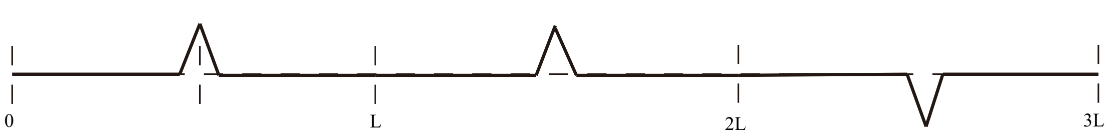
In this case, the function $f(x)$ in the range $L \sim 2L$ should be the same as function $f(x)$ in the range $0 \sim L$:

$$f(2L-x) = \sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi(2L-x)}{2L} = \sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi x}{2L} = f(x) \tag{67}$$

While the function $f(x)$ in the range $2L \sim 3L$ should be the opposite of function $f(x)$ in the range $0 \sim L$:

$$f(x+2L) = \sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi(x+2L)}{2L} = -\sum_{m=1}^{20} B_m \sin\frac{(2m-1)\pi x}{2L} = -f(x). \tag{68}$$

Hence the shape of string:



Problem 6

a.

The boundary conditions for the electric field is

$$\vec{E}(0,t) = \vec{E}(L,t) = 0. \tag{69}$$

Hence the n-th normal mode is

$$E_{y(n)}(z) = \sin\frac{n\pi z}{L}, \tag{70}$$

$$E_{y(n)}(z,t) = \sin\frac{n\pi z}{L}\cos\frac{n\pi ct}{L}. \tag{71}$$

b.

The standing wave can be decomposed to:

$$E_{y(n)}(z,t) = \sin\frac{n\pi z}{L}\cos\frac{n\pi ct}{L} = \mathrm{Re}\left[\frac{1}{2}e^{i\frac{n\pi}{L}(z-ct)-\frac{i\pi}{2}} + \frac{1}{2}e^{i\frac{n\pi}{L}(z+ct)-\frac{i\pi}{2}}\right] \tag{72}$$

For a traveling wave component with unit wave vector $\hat{k}$,

$$\vec{B} = \frac{1}{c}\hat{k}\times\vec{E} \tag{73}$$

Hence

$$\begin{aligned}\vec{B} &= \mathrm{Re}\left[\hat{z}\times\hat{y}\frac{1}{2}e^{i\frac{n\pi}{L}(z-ct)-\frac{i\pi}{2}} - \hat{z}\times\hat{y}\frac{1}{2}e^{i\frac{n\pi}{L}(z+ct)-\frac{i\pi}{2}}\right] \\ &= \frac{\hat{x}}{c}\cos\frac{n\pi z}{L}\sin\frac{n\pi ct}{L}\end{aligned} \tag{74}$$

c.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = -\frac{\hat{z}}{\mu_0}\sin\frac{n\pi z}{L}\cos\frac{n\pi z}{L}\cos\frac{n\pi ct}{L}\sin\frac{n\pi ct}{L}. \tag{75}$$

$$\langle\vec{S}\rangle = 0 \tag{76}$$

$$U_E = \frac{1}{2}\epsilon_0\vec{E}^2 = \frac{1}{2}\epsilon_0\sin^2\frac{n\pi z}{L}\cos^2\frac{n\pi ct}{L} \tag{77}$$

$$\langle U_E\rangle = \frac{1}{4}\epsilon_0\sin^2\frac{n\pi z}{L} \tag{78}$$

$$U_B = \frac{1}{2\mu_0}\vec{B}^2 = \frac{1}{2\mu_0 c^2}\cos^2\frac{n\pi z}{L}\sin^2\frac{n\pi ct}{L} = \frac{1}{2}\epsilon_0\cos^2\frac{n\pi z}{L}\sin^2\frac{n\pi ct}{L} \tag{79}$$

$$\langle U_E\rangle = \frac{1}{4}\epsilon_0\cos^2\frac{n\pi z}{L} \tag{80}$$

Note that

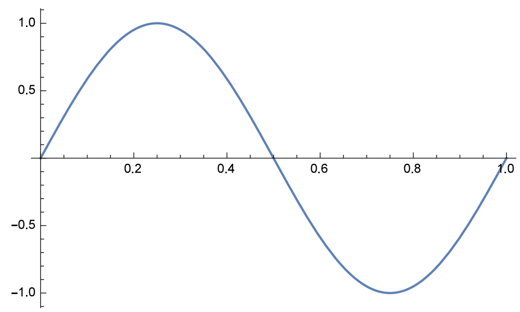
$$\langle U_E + U_B \rangle = \frac{1}{4}\epsilon, \tag{81}$$

which is evenly distributed.

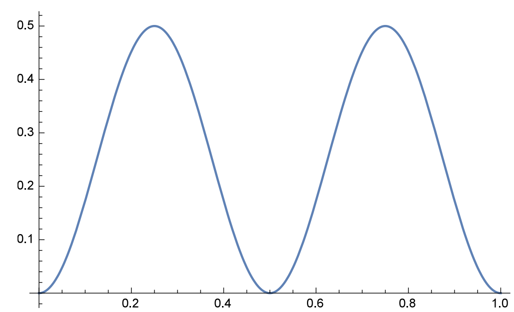
The amplitude of the normal mode is set to be 1 (without unit).

d.

The plot of E_y:

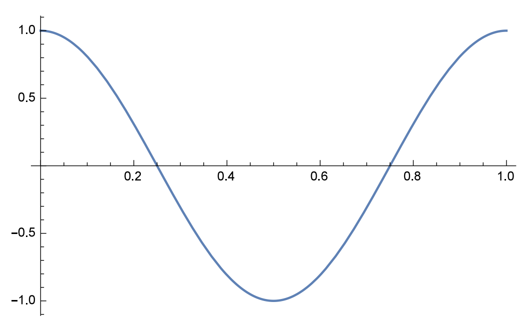


The plot of U_E:

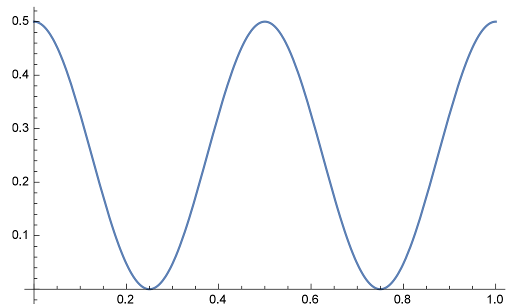


$\vec{B}$ and U_B identically vanish at $t = 0$ we chose.

However, at another time when $\vec{B}$ and U_B does not vanish, the plot of B_x:



The plot of U_B:


0.5
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1.0