YEN-JIE LEE: OK, so welcome back to 803. Happy to see you again. So today, we are going to continue our discussion of dispersive medium. And there are two questions we are going to ask in this lecture, and we will answer also these two questions in this lecture.

So just to warn you in advance, this lecture will have a lot of mathematics, so fasten the seatbelt and follow me. And stop me any time you don't feel like you know you understand something.

So let's get started. OK, so today, we are going to talk about phenomena related to dispersion. And last time, we started a discussion about how to send information from one place to the other place right. So what we should be before was to send square pulse.

So if I do have a machine which can produce a square pulse, then I can define something like this. So over some ratio, which I set, I can actually separate 0 and 1. So if I have a pulse which is actually having an amplitude greater than some threshold and I say, OK, I've got 1, and if it's actually below some straight line, say, OK, I've got a 0. And with that way, we actually can send information from one place to the other place. So that sounds really nice.

However, if we work on a dispersive medium, which is really very common-- for example, light and gas is actually-- not all the lights with different wavelengths are traveling at the same speed, and also, as you've seen before in the p-set, deep water, and also the strings, considering a realistic string with stiffness, et cetera, et cetera-- to the wavelength of the input wave is going to affect the speed of this travelling wave.

So in short, the speed of the wave propagation in a dispersive medium will depend on the wavelengths of this wave. So that brings us a lot of trouble because, for example, here we are trying to send a Gaussian pulse through the medium, but after a while this pulse actually becomes wider and wider because of the dispersion. Because all the components with different wavelengths which actually construct this narrow pulse, actually are traveling at different speeds.

Therefore, if you wait long enough, all the different frequencies, or all the different frequency harmonic waves are travel at different speeds, therefore you get the, dispersion, which results in a much wider pulse in the end. And at some point, this pulse is going to be really wide, such that it's actually going to be very difficult to separate 0 from 1.

So that's the problem. And we also did some simulations with computer. We do see this behavior also in our computer simulation. If I put in triangular pulse and allow it to evolve, and like what we did before, we assume that there's a stiffness in this string system. And you will see that, OK, as a function of time, this part is now longer a triangular shape, but you have a very complicated structure.

So that is actually a problem we are going to try to solve today. And during that discussion last time, in the lecture, we also introduced dispersion relation omega k and also tried to overlap two travelling waves with similar wavelengths.

And that would give you beat phenomenon. That probably doesn't surprise you any more. As you can see from this example, you have the beat phenomenon, and you can see the amplitude is actually variating slowly, the function of position.

And if you follow the red point, which is actually associated with one of the peak, in the structure called carrier, OK, it's actually moving at the phase velocity, the P, which we introduced last time. The formula for BP, which is actually the speed of harmonic, oscillating travelling wave is actually defined as vp equal to omega over k, and the green point, which actually always at the minima of the distribution, which is actually associated with the speed of the envelope.

You can see that, indeed, it actually can move at different speeds. It depends on the dispersion relation omega as a function of k you have in this system. And we call the speed of these envelope, which we construct from these two overlapping travelling waves to be-- we call it group velocity. And the definition of group velocity is vg equal to d omega dk. So that's what we have learned last time.

OK, you may ask, OK, what do I mean by group velocity? And can I use it beyond what we have done for the beat phenomena. But what do I mean by group velocity? Is that really useful, and it's actually which part of the structure I was talking about. So in that case of two overlapping progressing waves with similar length or similar frequency, when we see that the group velocity actually present the speed of the envelope, right? Can we actually learn something more general about group velocity?

The second question which we are asking is, OK, now we have this problem of dispersion. This square pulse is going to be something which is really wide after some period of time. So that's clearly a problem, and how do we actually solve this problem, and how do we actually send information like, for example, music over a large distance from one place to another place. So that's essentially what we're going to try to understand today.

So let's start with an infinitely long string. And this string is actually very long, and this began from here, and it goes to some place which is really, really far away. And, of course, as usually, I can actually hold one end of this string and shake it a bit, then I can actually create some kind of pulse which is going to travel along this string towards a positive x direction. In this case, I defined the x coordinate will be pointing to a right hand side, and thus the positive direction.

So of course I can hold this string, and I just shake it, and I would prepare a pulse on this medium, which is a string with constant tension. So I can describe the motion-- you can describe the motion of Yen-Jie's hand by a function. So you can say, OK, Yen-Jie is somehow doing a really nice job and oscillating at constant frequency.

Like I can say, OK, Yen-Jie is shaking this thing to produce a harmonic wave, for example. And that, I can actually describe the motion of the hand by f of t. That's very good. And from what we have learned in the last lecture, we've found that, basically, waves, harmonic waves with different frequency, or with different wavelengths, are traveling at different speeds.

Therefore, we would like to actually decompose the motion of Yen-Jie's hand into many, many harmonic waves-- then attack them one by one, to follow them one by one, then I can solve this problem. So that's actually what we are going to do.

And that will involve some math, which we would follow from the math department. And before that, I would like to introduce the imitation first. As I said f of t is actually the displacement as a function of time as x is equal to 0. So basically, I'm holding this string, and I move things up and down, so that, actually, I move this string away from the equilibrium positive, which is actually y equal to 0.

Then what is going to happen? What is going to happen is that I'm going to produce some kind of pulse, and this pulse, I can actually describe it by a function, which is psi x and p, this psi is actually describing the displacement as a function of x, and as a function of t.

Apparently, if you put x equals to zero, then you go back to f of t, right? Basically that's the idea. OK. So what we have learned before we introduce dispersive medium is that, if I have a non-dispersive medium, OK, if I have a non-dispersive medium, then things are pretty simple because omega over K is actually a constant, which is the phase velocity, vp.

And omega is actually just equal to vp times k. That means, no matter what kind of wavelength we are talking about, no matter what kind of angular frequency we are talking about, harmonic progressing wave is going to travel at the speed of Vp. No matter what's the frequency, or what's the wavelength.

So that makes our life much simpler when we work on non-dispersive medium. In this case, if I have a non-dispersive medium, then psi would be equal to-- maybe I write it here-- if I have non-dispersive medium where, no matter what kind of frequency, the speed of the harmonic traveling wave is a constant, which is actually Vp, I can write down psi x t to could be equal to f of t minus x over v.

Just remember f is actually describing how I shake one end of the string, and, basically you can see that ha! What is happening is that my hand is actually generating the shape of the pulse as a function of x, as a function of time, and it can be described by a really simple formula here.

So this is actually really nice for non-dispersive. As I introduced before, when we talk about dispersive medium, then, if I go to dispersive, omega is actually a function of k, and can be a non-linear function. So what does that mean? That means, if I evaluate vp, which is actually the phase velocity, which is the formula there, this is going to be omega of k divided by k.

That means BP is going to be a function of k, the wavelength-- wave number. It's not going to be a constant in general-- unless omega is actually equal to vp times k, in general, vp can actually be some quantity which is variating as a function of k. OK? Then we have trouble because that means, when I produce progressing wave from the left hand side end, it's actually made of many, many harmonic waves, right, with different angular frequency.

So I can shake this like [MAKES NOISE], different speed. And I can always decompose the motion of Yen-Jie's hand into many, many harmonic waves. The problem is, all those harmonic waves are going to be travelling at different speed. How do we actually describe this?

So that's the trouble. And I was really frustrated when I think about this problem, and my friend from the math department said, hey, we have solved this problem a long time ago.

[LAUGHTER]

So this is not the problem anymore. And I say, oh, what is the idea you're talking about? And they actually told me that you should use Fourier transform to attack this problem. OK?

This is the idea. The idea is that I can now write down f of t, which is motion of Yen-Jie's hand, and this can even returned as a superposition of infinite number of waves. I can integrate from minus infinity to infinity, t omega, which is the angular frequency. And each contributing wave has an amplitutde associated with it, which is, as you see, is a function omega. And the actual wave is actually written in terms of exponential minus i omega t.

So now, let's actually you look at this thing really carefully. What am I doing? I am saying that I, now, can shake one end of the string up and down according to my will. And, if I do this for a long time, I can actually describe the motion of my hand by infinite number of harmonic waves, which is actually kind of like exponential i omega t describing the frequency of these waves, and each of them got associated amplitude.

And you may ask, OK, wait a second, you call this Fourier transform, and I have learned that before, but I learned a different version. I learned a version of cosine and sine? And what is going on?

Actually, they are all the same. No matter what you do, you can actually also do that with cosine and sine, but what I actually found is that it's actually easier to deal with exponential functional form You can always write exponential i omega t in terms of sine and cosine and absorb the i into a c omega.

Basically, these things are identical between these two forms of this answer. So therefore, in this lecture, I'm going to stick with this functional form. OK, any questions?

STUDENT: We don't include dx in the [INAUDIBLE]?

YEN-JIE LEE: Not yet. We are going to include that. Because, for that, in order to actually-- OK, so now I actually decompose the motion of my hand into many, many waves-- which should be or is say it many, many oscillation with different frequencies.

So I actually describe the motion of my hand infinite number of oscillation with different frequency. And the trouble we are facing is that all those oscillations are going to be charged travelling at different speeds because of the dispersion relation.

Therefore, what I am going to do afterwards is to show you that, OK, I can write down the functional form for psi in this general case. So for that, that's actually what I'm going to do now. So now, I would like to know what would be the psi xt, which actually the position of the string as a function of x, and at some specific time equal to t.

And that can be written as, I do the an integration from minus infinity to infinity over frequency omega, and I have the usual amplitude associated with the angular frequency omega, and the exponential i omega t-- minus i omega t because that's the convention I'm using here-- and I say, OK, plus ik, which is a function omega-- x.

So now you can to see that what I'm doing here is that I am now progressing, I am making infinite number of progressing waves. Each of these exponential functions is a progressing wave with angular frequency omega.

And why do I write k as a function of omega x here? It's because they are going to be travelling at the speed of omega over k. Therefore, I need to actually put k here, and this k is actually-- this k is actually not the independent parameter.

It's actually a function of omega. So we can see that, here, we do an integration over omega from minus infinity to infinity-- for each omega you can actually find the corresponding k, right? Because of the dispersion relation. Because omega is a function of k, therefore you can always solve the corresponding k, right?

Then you put it there? Because you are now trying to propagate how many waves with different angular frequency at different speed-- then we are done. That looks like a wonderful solution, and we can actually see how it works for our purpose. Any questions? All right.

So that's really nice. And I can now do a really simple test to see if this really works. Let me try a very simple case. OK, a spatial case. If I now go back to use this description to describe non-dispersive medium and see what will happen.

Now my k as a function omega is actually rather simple. It's actually omega over vp-- according to the dispersion relation here. I can solve k, as I was mentioning, with these dispersion relation formula. And then I can conclude k as a function omega is omega over vp.

Then I can now put that into this equation, and I'm going to get psi x of t-- this would be equal to minus infinity to infinity d omega, c omega, exponential minus i omega t minus omega over vx.

And we can actually take omega out of this, minus infinity to infinity d omega c omega exponential minus i omega t minus x divided by v. And you can see that, huh, indeed, this is actually ft minus x over v. OK. I'm dropping the vp here.

This should be vp all over the place. So you can see that, now, if I have solved the k as a function omega, and I plug it in in this special case, which is non-dispersive medium, omega over k equal to vp, then I really calculate this integral, then I can quickly identify that-- huh, I can write the functional form in this way.

And this is actually really familiar to me because that's actually using this definition, f is actually equal to integration minus infinity to infinity, d omega, c omega, exponential minus i omega t. If I replace t, by t minus x over vp, then I'm done. So I have evaluated this integration, which is actually just f t minus x over vp.

So that's exactly what guessed from the beginning. So if I have a non-dispersive medium, then psi xt will be equal to this function. So that gives us some kind of confidence that, OK, at the easy case, it works.

All right, so that's very nice, all sounds very good. But wait a second. How do I actually extract this c, which is a function of omega? I'm troubled because this is an infinite integral from minus infinity to infinity. And that means I have infinite number of constants, which I have to determine the c omega. How do I actually do this?

So that is another point which I would like to discuss before we actually go ahead and really use this function for the dispersive medium case. So how to we actually extract c as a function of omega?

So for that, we really need to employ a few uses for formula, which are actually documented here. How many of you actually have not heard about delta function before? OK, a few of you actually have not heard about delta function.

So what is actually your delta function? This is a delta function. So a delta function is actually a notation which actually shows you a function, which is should only be non-zero, at x equal to zero. And the x equal to zero, the size of this function as you're going to infinity, and on the other hand, all the other points at x not equal to zero, the delta function is equal to zero. So that's actually the kind of function I was talking about. And the area of this function, if you're doing the equation over minus infinity infinity over x, the integration of these delta m the area is actually 1.

So that is actually the kind of function. So essentially, it's a really, really narrow function, OK, very narrow, very narrow, very narrow. But the area is finite, which is why. So you can have a square. You can actually start with a square pulse, or square function, and you can actually make the width of the square narrower, smaller and smaller and smaller, go to 0. Then what you are going to get is essentially the delta function. That's actually how we understand this delta function.

All right, really quickly. And also, we would like to use a few formula which are documented here. So if I do an integration from minus infinity to infinity, exponential i omega minus omega prime, t over the t which is integrating over time, t, here. And then divide the whole formula by 1 and over 2pi. What I'm going to get is a delta function, which is a delta function which is omega minus omega prime. So that means when this delta function formula tells us that omega is equal to omega prime, then this function is actually going to infinity. And only when omega equal to omega prime, this function is not zero. Any other place, this function is always zero.

And this strange integration should give you this delta function. So that's the first thing which we will use, was one useful formula. The second thing which what just I talked about, if I do an integration over minus infinity to infinity, delta x dx, then basically you get 1. The third one is actually kind of interesting. Let's take a look.

So if I do an integration over from minus infinity to infinity, delta function x minus alpha. Let's look at this delta function first. This function is only non-zero when x is equal to what?

AUDIENCE: Alpha.

YEN-JIE LEE: Yeah. When x is equal to alpha, only when that happen, this is actually non-zero. If you multiply this delta function to some function which is f of alpha, and integrate over alpha from minus infinity to infinity. And that means that when alpha is equal to x, or x equal to alpha, this integration give you non-zero result. All the other ways, you will get zero.

The interesting thing is that if you do this integration, what you are going to get is that OK, when I integrate over alpha, only when alpha is equal to x this thing is non-zero. Therefore what you are going to get is, you get only one point of the width, which is actually f of x. So that's the intuition about this formula. That's just fine. Any questions related to those formulas?

AUDIENCE: [INAUDIBLE]?

YEN-JIE LEE: Hm?

AUDIENCE: [INAUDIBLE]?

YEN-JIE LEE: Yeah, this is actually pretty complicated, so it would take a few 10, 20 minutes to explain that. But let's just take the words from the math department-- we trust them. All right, so once I have those formula, I can now demonstrate you how I can actually track C as a function omega. So this is actually the goal, right? So don't forget why we are doing what we are doing, is to try to extract what is actually the C omega, so that we can actually finish this formula.

So how do we do that? So suppose, if I evaluate this. This function, 1 over 2pi, minus infinity to infinity dt, ft, exponential i omega t. If I evaluate this function. This is coming out of nowhere, right? So coming out of Yen-Jie's hand, maybe, I don't know.

Suppose if I evaluate this function, and now I have ft here, right? I can replace ft by these interesting formula. If I do that, then basically I get 1 over 2pi minus infinity to infinity dt, minus infinity infinity C omega prime, exponential minus i omega prime t. And the last is actually integrating over d omega prime.

So this is actually the f of t. This is actually f of t. I'm just replacing that formula into this integral. And then I have the rest, which is exponential i omega t. And of course, I can continue and collect all the relevant terms together. This is actually equal to 1 over 2i, minus infinity to infinity. I collect all the terms related to omega prime to the left hand side.

Basically what I get is C omega prime d omega prime. This is actually coming from here, except-- yeah, OK, it is actually coming from here. And I have another integral which is from minus infinity to infinity, this time integrating over delta dt. And I have dt here, exponential i omega minus omega prime t. So basically I'm collecting these two terms together.

They now become exponential i omega minus omega prime t. So basically, no magic happened, but I'm just re-writing things and we are arranging things from this formula to that formula. Then if we look at this formula, this formula here, and the formula sheet we have. 1 over 2pi minus infinity to infinity to this integration over t, exponential i omega minus omega prime t. That will give you delta function, which is delta omega minus omega prime.

Therefore, I can continue this calculation here. Thus it's going to give you minus infinity to infinity. I identify this part, this part, and this part, to be the delta function. Therefore, what I get is minus infinity to infinity, C omega prime, delta omega minus omega prime, d omega prime. Am I going too fast? Everybody's following?

So you can see that what we have been doing, I use this formula coming out of nowhere. I replace f by the formula I was writing there. And then I collect the terms I like together. That's all I did. And then I found, aha! One part of the formula is actually the delta function.

Then I put the delta a function here. And then finally, I use the third formula here, which I have related to delta function, and I found, aha! If I do this integration, I know how to do this integration even without knowing the structure of C. This is actually just changing the omega prime to omega. So that's actually what this integration actually does.

Therefore, I get C omega. Look at what we have done. What we have done is that, we have proof that this formula coming out of nowhere, to be a continuous version of mode picker. You remember the fourth year decomposition before? You were using the orthogonality of the sine function, and I can do some kind of fancy integration to actually extract a m from one of the-- which is associated with one of the normal mode, right?

What we are doing here is actually a continuous version. Now omega is actually continuous. And I'm now using the orthogonality of the exponential function. If I do this integration, that will only give you non-zero value when omega is equal to omega prime. It's exactly the same thing, right?

Then I can construct an integration like this. And now will give you the redoubting C as a function omega, which is like the amplitude of one of the associated harmonics exponential i omega t. So in short, from this exercise, we have shown you that C of omega can be extracted using this formula 1 over 2pi, minus infinity to infinity dt, f of t, exponential i omega t.

That's actually how we actually can determine all the amplitude associated to a specific exponential function. Any questions so far? OK, so if no question, then we can actually continue. So let's actually go back to the original question, which we were posting. So we have a problem related to the transmission of information.

So this is actually where we got started. If I send a square pulse on a dispersive median, then I have some trouble, which is that this pulse is going to disperse and become wider and wider. It's changing as a function of time, as a function of distance it travel. That's not cool. All right, so therefore what I am going to do is this.

There was a very smart idea which were discovered long time ago, during maybe World War I, and widely used in World War II, which is the AM radio. What is AM? Is actually amplitude modulation radio. This smart idea is the following. I will describe it before we take a break. So this smart idea, AM radio is the following.

If I have some kind of information which is fs t. s here means signal. If I have some kind of information I would like to send, I can send it by oscillating one end of the string. And this is what I want to send. And there are two ways you can send this fs function.

The first one is actually what did before, I send it directly. I just said OK, if I want to send this function, then I just oscillate the string according to the functional form. Yen-Jie just have to be really careful, right? So that you can send this function. And that fails miserably.

Why? Because all the components which actually produce the fs, in this case the square pulse, all those components are travelling at different speed. Therefore, the information will never get there, because of the dispersion. So now what should we do instead? Instead of doing sending fs as a function of t directly, what you could do is that I can now send f of t, which is equal to f of s t cosine omega0t.

Where omega0 is a very, very large number. And basically, look at what we have been doing. So that means I, instead of sending fs directly, I send fs, but modulated by a really high frequency function, cosine omega0t. And this will work. And you will only know that after we come back from the break, which is twenty first. Let's take five minute break. And if you have any questions, you can actually ask me here.

So we will continue the discussion about AM radio. So before the break, actually we introduced this one possible solution to solve this dispersive median problem, is that I can now actually send instead of fs as a function t, which is actually the signal I want to send, I could send fs, but multiplied by cosine omega0t.

If I assume that fs is some really slow function, slowly varying as a function of time, compared to cosine omega0t. Cosine omega0t is a really fast function, oscillating up and down like crazy, really fast. If I multiply fs by this function, what is going to happen? We are going to show you that actually that means I am going to only have non-zero C function, or a large contribution of C, in a very thin middle range of omega.

So we'll show that. So in a typical case, fs is really slow, which is like, for example, my sound, et cetera, in the label of one kilohertz. And you can actually design a system which will actually multiply this fs by cosine omega0t. Omega0 can be as fast as 1.1 to 30 megahertz. If you do this calculation, then you will find that OK, the range of omega, with sizable C omega is small.

It's roughly equal to omega0 minus omega s, to omega0 plus omega s, where omega s is the typical frequency in your signal. And the omega0 is the typical frequency of-- the frequency of your cosine omega t term, which is actually, later, you will recognize this as carrier. So what I want to say is that if I do this trick, what is going to happen is that the range of omega, which you have sizable contribution from C omega-- C omega is the associated amplitude, associated amplitude.

It's going to be confined to a really small region from omega0 minus omega s, to omega0 plus omega s. So that's the trick which actually makes this problem solvable. How do we know this? That is because, if I now, for example, I send fs equal to cosine omega st. If this is actually the signal which I would like to send, just a harmonic wave, then what is going to happen is that I'm going to get ft is equal to cosine omega st cosine-- so this is actually multiplied by cosine omega0t.

So I have cosine omega0t here. I have cosine multiplied by cosine. Therefore I have the question which I prepare here, the formula, of cosine alpha times cosine beta will be equal to the functional form. There's a remainder, therefore I can now write it as 1 over 2 cosine omega s minus omega0 t plus cosine omega s plus omega0.

You can see that when I actually multiply two cosine functions together, then what I get is actually the omega0 minus omega s. You can actually put the minus sign there, it didn't matter. And cosine omega0 plus omega s times t. So therefore, you can see that the frequency, there are only two frequencies which contribute to this C of omega, which is actually these two frequencies.

So that is actually why, if you do this trick, you actually try to modulate your slow signal function by a fast carrier frequency. Then you are going to confine the effective range of omega into a very small range. Why is that useful? That's actually what I want to answer to you.

Suppose I have this crazy dispersion relation, which is omega as a function of K. You can graph it, and suppose it looks really crazy like this. And if I set my carrier oscillation frequency to be omega0, and that will give you a corresponding wave number which is K0. I hope you can see it. That's the corresponding K0.

Before we actually multiply this function, it's a slow function. It's not exactly one cosine function. So if you just have a cosine function harmonic wave, then you don't really need this trick, because it's actually going to be traveling at a speed of some constant speed. It's a harmonic traveling wave. But if this is actually a slow function, but not really a single harmonic wave, then what is going to happen is that you are going to need a wide range of K value or omega value to describe fs.

Then you are in trouble because now, all the waves with different wavelengths are going to be travelling at different speed. Then you have this dispersion problem. On the other hand, if I multiply this function, this slow function, by a fast oscillating function, I am confining the effective range of omega into this small box, which is actually between omega0 minus or plus omega s.

This is actually omega0 plus minus omega s. This is actually the range of the possible omega, which contribute to this resulting f of t. Therefore, the behavior of this function is actually much easier to understand. So with that given there, suppose now I have a large difference between-- suppose I have a large difference between omega s and omega0. Then I can actually focus on a very small range in this dispersion relation diagram.

Then I can write omega as a function of K, the dispersion relation equal to omega0 plus K minus K0, partial omega, partial K. Evaluate it at a equal to K0, plus higher order term. Basically I can do this Taylor expansion. And maybe it surprised you, you can immediately identify, ha! This is delta d omega dk, is the group velocity.

Suddenly it show up in the Taylor expansion of the dispersion relation. so if I focus on the region which is around omega0, then I can actually re-write omega in this functional form.

Omega is actually equal to omega0 plus K minus K0 times Vg, which is the group velocity. Suppose this is happening, then now I can actually go ahead and really calculate the functional form for f of t.

So suppose I have this definition of t. The definition of t is equal to fs t times cosine omega0t. Or say I can actually write it in a complex form, instead of writing it in a cosine omega0t functional form, I can write it in exponential functional form. Exponential minus i omega0t, which is actually more convenient for the discussion.

So what is going to happen if I actually do this calculation? Then basically that's one example signal which I would like to send on the slides. So if I am trying to send a progressing harmonic wave, then after multiplying by this exponential i omega0t function, or a cosine function, basically you get something which is actually oscillating really fast, which is actually the AM signal we are trying to send through this media. So we can actually identify, this is actually the structure of this, actually the carrier. And this signal become the analogue, in analogy to what we actually have discussed for that beat phenomenon case.

So now, if the omega range is really small, then I can actually write this down. Write a functional form of omega in this functional form. Or I can actually take out the kVg term, and the rest is actually going to be something like some constant a, where a is actually equal to omega0 minus Vg times K0. So basically I'm just taking out this K term here, and this become this term.

With this formula, I can solve what would be the functional form for K, as a preparation for what I'm going to do later. So K can be also expressed as omega over Vg. So basically I just solve the K plus b. b is actually just some constant. Just do it for convenience, I can write b equal to K0 minus omega0 divided by Vg.

What we learned here is that if the range of effective omega is really small around omega zero, then the relation between omega and the K becomes a linear function. Of course, it's still not like the case for the non dispersive median, where omega over K is a constant. But at least it becomes a linear function, which is actually much nicer.

So finally, with all those preparation we have done, we would like to show one important consequence. So what we are trying to do is to show that psi xt. Now I send, I oscillate the median, the string, by this f of t, which I designed there. ft is actually fs times exponential i omega0t. That's actually designed there. I would like to show that the resulting amplitude will be equal to fs t minus x divided by Vg, exponential minus i omega0t minus K0x.

Of course I need to take the real part of this in, to go back to the real axis. Basically I dropped the i sine omega t contribution. So this is actually what I want to show. Before I go through all those math, let's do get the conclusion which we would like to draw, before we actually really go through the math. The conclusion which I would like to draw is that, OK, this is actually my analogue. My analogue is going to be travelling at the speed of Vg, which is the group velocity.

That's the conclusion which I would like to draw from this exercise. And this thing is actually cosine omega0t minus K0x. Therefore, this is actually a harmonic wave. The carrier is a harmonic wave travelling at Vp equal to omega0 divided by K0. That's the kind of conclusion which I would like to draw from this exercise. Any questions about what we have discussed so far?

OK, then really you have to hold tight and follow me really, 100% focus, because this is actually a complicated calculation. So now what I can do is, now I need to express my fs in terms of C. So I do integration from minus infinity to infinity, d omega, C omega, exponential minus i omega t. So basically I can write my f of s in a functional form, which we introduced before. Then my f function is actually equal to fs times exponential i omega minus i omega0t.

So that's actually what we defined there. And this would be equal to minus infinity to infinity. I do this integration number, omega C omega exponential minus i omega plus omega0 times t. So there's nothing special, I just take my expression for fs, multiply that by exponential minus i omega0t. Then that's actually what I get.

So since this is actually integration over omega from minus infinity to infinity, therefore I can always have the freedom to shift the origin. So that means f of t can be returned as minus infinity to infinity d omega C omega minus omega0 exponential minus i omega t. Then we can see that is fix a relation between C of the f function, and the C of the fs function. So so far, everything is exact. I haven't made any approximation so far.

So now, I can take this function and propagate that to all x. In other words, I can now take this ft, and write down the psi as a function of x and t. So that means all the different components are traveling at different speeds. So basically, I can write it down like d omega C omega minus omega0, exponential minus i omega t, exponential ikx. Kx k is actually a function of omega.

Any questions? So that's actually just identical to what we actually have done before. So now I can go from f of t to sine, if you are following me. So until here, everything exact. You have all the problems you have, like you know this dispersion essentially, because all the little components, as you can see here, can be travelling at different speeds.

So now, what I could do is that if I assume that C omega is only sizable at the small range around, it's only sizable around omega0. If now I take this assumption and propagate into this formula, then I can write this psi function roughly like minus infinity to infinity d omega C omega minus omega0 exponential minus i omega t exponential i. Now I can take the formula which I actually did an approximation, around omega0.

Around omega0, K can be returned us omega over Vg plus b. This is actually where I take the approximation. Only consider the first order in the Taylor expansion. So you can see now here, it's not exact anymore. But now I write approximate function of form for K omega. So what I'm going to get is omega over Vg plus b, multiplied by x. Any questions?

Now I have the approximation. And of course now I can gather all the terms related to omega together. I'm getting minus infinity to infinity d omega C omega minus omega0 exponential minus i omega t minus x over Vg, exponential ibx. So basically, I am merging this term and that term. This term and that term will give you this term. And what is essentially the rest is the exponential ibx.

We are almost there. So now I would like to use this board, so I need to erase that. So now I continue from here, and I can now again, I can again change the origin of this infinite integral so that this can be written as minus infinity to infinity, d omega C function of omega, exponential minus i omega plus omega0, t minus x divided by Vg, and exponential ibx. So what I come from this board to that formula, if you are following me we are almost there, because I am changing the origin again, so that omega minus omega0 becomes omega, become a new omega.

Is everybody accepting this fact? And that means the original omega will become omega plus omega0. I'm trying to go really slow, so that everybody can follow. I hope you are following. All right, then now I can actually redistribute, arrange all those terms and the magic will happen. So that means rearrange all those terms, minus infinity to infinity d omega C omega exponential minus i omega t minus x divided by Vg, exponential minus i omega0t, exponential i omega0 over Vg plus b x. So basically, there's really no magic.

What I'm doing is really to rearrange all those terms, so that this term is actually rearranged so that it's now omega times t minus x over Vg. It's an independent exponential term. And I actually extract this term times t to be returned here. I'm just rearranging things, OK? I'm not changing anything.

And finally, I can merge this term and that term, and become this function field. I can immediately recognize that after this rearrangement, this is just re-writing the formula, putting all those terms in different place. Of course, you can actually review this part of the lecture in the lecture notes later. But basically, we're not doing anything fancy but rearranging things over in different place. Then I can actually quickly identify what I am trying to integrate.

So this integration is over omega. Therefore all those terms are now related to omega. Therefore, they are just some terms which are sitting there, they don't participate. And if you focus on this part, what is this? If you compare that to the original equation of which I have here. If you compare that to the original fs equation here, you can't immediately identify that actually that's a function of fs. Originally this function fs is a function of t.

And I'm going to that board now. This is actually fs with t minus x over Vg. Surprisingly simple. Now let's look at the right hand side, this mass here. This is actually K0, which actually you cannot see anymore. It's in the back of this board. And then if you combine these two terms, basically what you get is exponential minus i omega0t minus K0x.

So look at what we have done. I got started with this Fourier transform functional form of fs. I multiplied fs by cosine omega0t and go to the complex notation. It becomes exponential minus i omega0t. If I multiplied that, I get my f function, which is like this. You get additional term there.

I rearrange things and change the origin, and I can rewrite ft in this functional form. And I can have a relation between the C related to fs to the C related to f of t. I propagate ft over the full space, and attain my sine, which is the amplitude as a function of place and the time. Until here, everything is exact. Then I have introduced assumption, which is C of omega is only sizable, only contributing, around omega zero, therefore I can do approximation form for the K function, which is this functional form.

Then I just do the integration. Then I found that, interesting! This side is-- you should be taking the real part of this thing. This side have two components. The first component is fs, which is the original signal you put in, the signal you want to send. It's actually progressing at the speed of group velocity. So now you understand what this group velocity means.

That's the speed of the signal you want to send in the AM radio. And this thing is actually modulated by exponential function, which is actually the propagating at the speed of Vp, equal to omega0 over K0. So the carrier still, after you actually include many, many terms contracting the f function, the sine which is the amplitude, the trick is that only the omega value around omega0 contributes.

If that happen, then you can see that there are two structures actually propagating at different speeds, and that you can actually understand the structure independently. That means your signal will not be distorted if you're sending it this way. But the difference is that the speed of the signal you are sending is actually at the speed of group velocity. That is actually the amazing fact which actually enables us to send signal over thousands and thousands of miles away from the source.

So what is actually done is actually that, suppose you have some kind of radio station. You can send the radio, and the radio will go over the place, and got refracted by atmosphere-- the atmosphere on Earth. Got refracted, and the receiver from some place which is really distant from the source can still see it without any dispersion, as we show here. And it's actually going to be propagating at the speed of group velocity.

So you may not actually believe that. How about we do a simulation like what we did before with MIT wave? So this is actually the example which we did last time. We have an nit wave. We can compose that into many, many pieces. And then see how it evolved as a function of time. This is actually without dispersion, therefore everything is perfect. So now I would like to introduce some excitement there.

If I have dispersion, like 0.1, alpha is equal to 0.1, and see will happen. Then just a reminder that things will not go super well. Wait a second, what am I doing? This is actually still without dispersion. Sorry for that. It should be-- OK, so let's take a look at the triangular case.

This is now with dispersion. And you can see that as a reminder as a function of time, the shape of the signal which you would like to send is actually changing as a function of time. And after a few thousands of miles, you will not even recognize the original structure we put in. So that's the trouble we are actually facing. You can see that it's getting wider and wider et cetera.

So now what will happen if I send this kind of signal. This is a signal which you have some kind of shape. You can imagine that there's sounds kind of analogue. And I am now doing the calculation to actually map all the individual components. And now I'm going to propagate through the median. And the blue is the original non-dispersive median situation. And the red is actually the propagation in a dispersive median.

You can see the propagation in a dispersive median is faster, because alpha is actually larger than one-- larger than zero. So it's actually, in this case it's 0.1. And you can see that the red cosine omega0t modulated signal is progressing, and the shape of the analogue is not changing. You can see that, right? So it's very different from what we actually see before with a single triangular pulse.

Now you can see that, ha! Only when n gets very large, I start to be able to feed all those little structures. That means the end value, or say the omega value, which I need is you really narrow, a very narrow range, which will actually match with what we have been doing. And now I start to propagate all those things. And you can see that the red is actually traveling faster than the blue, which is what we expect. And you can see now, in the instance they actually overlap each other, you can see that envelope, the shape of the envelope, is still the same.

It's exactly what we actually printed. And that actually brings me to the end of my lecture. We have on understood how the AM radio actually works. And next time, we are going to talk about uncertainty principles.

What the hell? What happened? And believe me, they are actually connected to each other. Uncertainty principle is actually highly related to wave and the vibrations. Thank you very much, and let me know if you have any questions.