YEN-JIE LEE: OK, so welcome back everybody. Happy to see you again. So today, we are going to continue our exploration and understand single harmonic oscillator. And this is actually a list of our goals. And we would like to learn how to translate physical situations into mathematics so that we can actually solve the physical problem, so we actually have and predict what is going to happen afterward. And we also sort of started this course by solving really simple examples, single harmonic oscillators.

And as a function of time, you will see that, for our next class, the next lecture, we are going to bring in more and more objects. And of course, more objects means more excitement also, in terms of phenomena, but also more complication on the mathematics. So we will see how things go.

And then after that, we are going to go through infinite number of oscillators to see what will happen. Of course, we will produce waves. That's very exciting. Then we'll do all kinds of different tricks to do those waves.

So what we have we learned last time? So last time we went over example, a simple harmonic oscillator. It says you have a rod fixed on the wall, and you can actually go back and forth. And we also introduced a model of the drag force, or drag torque, and that's actually proportionate to the velocity of the motion of that single particle.

And the interesting thing we learned last time is that we have three completely different behaviors if we actually turn on the drag force. The first one is on that damped. Damping is actually very small. Then we have the solution in this form. It's oscillating. The amplitude is decaying exponentially. As we make the drag force larger and larger, you will pass a critical point, which actually give you a solution, which you don't have oscillation anymore. The cosine disappeared.

Finally, if you actually put the whole system into water or introduce something really dramatic-- a very, very big drag force-- then you have overdamp situation. And there you see that the solution is actually a sum of two exponential functions.

So this is actually the one equation which actually works for all the damped situation we discussed up to now. And this is actually the map. Basically, if gamma goes to zero-- gamma actually controls the size of the drag force. Then we got no damping. Then you have a pure, simple harmonic motion.

And as we increase the gamma, then you get see that the behavior is changing as we increase the gamma. So you can see that we can use a quantity, which is called Q. Q is actually defined as a ratio of omega at zero, which is basically the natural angular frequency of the system. And gamma is a measure of how big the drag force is. If we make a ratio of this to quantity, then you'll see that, at Q equal to 0.5, it reaches a critical point, which actually the behavior of the whole system changed. And you can see that the oscillation completely disappeared. So that is actually what we have learned last time.

So what are we going to do today? We have been really doing experiments really with our hands, hands-on, right? So basically we will prepare the system. Then we release it. Then we don't touch it again and see how this system actually evolves as a function of time. So that's what we have been doing.

So today, what we are going to do is to start to drive this system. We can introduce some kind of driving force and see how the system will respond to this external force. So that is actually what we are going to do. And that will bring us to the situation of damped driven harmonic oscillator. So let's immediately get started.

So we will use the example which we went through last time as a starting point. So set example from the last lecture is a rod, which is fixed on the wall. And the lens of this rod is over two. And I define a counter-clockwise direction to be positive. And I measure the position of the rod by this theta, which is the angle between the vertical direction and the pointing direction of the rod. And we have went through with the math, and we got the equation of motion without external force, which is already shown on the blackboard.

So now, as I mentioned at the beginning, I would like to add a driving force, or driving torque, tau drive. This is equal to d0 cosine omega d t. So I am adding a driving torque. The amplitude of the torque is actually d0. And there's actually also a harmonic oscillating force, or torque, and the angle frequency of this torque is omega d.

And that means our total torque, tau of t, will be equal to tau g t-- this is actually coming from the gravitational force-- plus tau drag, which is to account for the drag force. So this time we are adding in a tau drag. So I'm not going to go over all the calculations on how did the right from the beginning to the end. But I will just continue from what we actually started the last time.

So if I have additional driving torque there, that means my equation of motion will be slightly modified. This time, my equation of motion will become theta double dot plus gamma theta dot plus omega 0 squared theta, and that is equal to d0 divided by I. This is actually divided by I because, in order to get the acceleration, I'll need to divide my torque by a movement of inertia of this system and cosine omega d t. This is actually the oscillating frequency of the driving torque.

And just a reminder, gamma is defined to be equal to 3b m l squared. And the omega 0 is actually defined to be square root of 3g over 2l. So as I mentioned in the beginning, this is actually giving you a sense of the size of the drag force. And the right hand side, the omega 0 is actually the natural angular frequency.

So, of course, we can actually simplify this by replacing this term, or this constant, by symbol. So the symbol I'm choosing is f0. And this is defined to be d0 divided by I. Therefore, I arrive at my final equation of motion-- theta double dot plus gamma theta dot plus omega 0 squared theta, and that is equal to f0 cosine omega d t. So I hope this looks pretty straightforward to you.

So this is our equation of motion you can see from this slide. So we have three terms in addition to the theta double dot. The first one is actually related drag force, or drag torque. The second one is actually related to so-called spring force. So that is actually be related to the spring constant or, because of the restoring force of the gravitational force. The third one is actually what we just add in as a driving force.

So one question which I would like to ask you is-- so now I bring one more complication to this system. So now I am driving this system with a different frequency, which is omega d. The question is, what would be the resulting oscillation frequency of this driven harmonic oscillator? What is going to happen? Well, this system actually follows the original damped oscillator frequency, omega, which is actually close to omega 0. Or what this system actually follows the driving force frequency. Finally, maybe this system chooses to do something in between. We don't know what is going to happen.

So our job today is to solve these equation of motion and to see what we can learn from the mathematics. Then we can actually check those results to see if that agrees with the experimental result, which is through those demos, OK?

So as usual, I have this equation of motion here. So one trick, which I have been using, is to go to complex notation, right? Therefore, I can now re-write this thing to be Z double dot plus gamma Z dot plus omega 0 squared Z, and that is equal to f0 exponential i omega d t. So basically, I just go to the complex notation.

And we would like to solve this equation. So in order to solve this equation, I make a guess, a test function. I guess Z of t has this functional form. This is equal to A-- some kind of amplitude-- exponential i omega d t minus delta. Delta is actually some kind of angle, which is actually to account for the possible delay of the system.

So if I start to try-- for example, this is a system, which I am interested-- I start to drive this system, it may take some time for the system to react to your driving force. So that's actually accounted for by this delta constant. And the amplitude is actually what we were wondering, what would be the amplitude. Therefore, you have some kind of a constant in front of the exponential function. And you can see that this exponential is actually having angular frequency, omega d. And that is actually designed to cancel this exponential i omega d t here in the drag force.

So now we can as you calculate what z dot of t would be equal to i omega d Z. Z double dot t will be equal to minus omega d squared Z. With those, we can now plug that back into this equation of motion and see what we can actually learn from there.

So basically, what I'm going to do is to insert all those things back into the equation of motion. And that is actually going to be like this. Basically, the first term, the double dot, you get a minus omega d squared out of it. The second term, gamma Z dot-- I have Z dots here. Basically, I would get plus i omega d gamma out of the second term. That's third term, I get omega 0 squared out of it. And that is actually multiplied by Z. And this is equal to f0 exponential i omega d t.

All right, and we also know from this expression Z is equal to A exponential i omega d t minus delta. That's the test function. So this is actually equal to A exponential i omega d t minus delta. So now what I can do is-- I have some constant in the front. Multiply it by exponential i omega d t. And now I can actually cancel this exponential i omega d t with the right hand side term. Very good. The whole equation is actually exponential free. Now I don't have any exponential function left. And exponential i delta is actually just a constant. So now this equation is actually independent of time.

So what I getting is like this-- basically if I multiply the both sides by exponential i delta, then I get minus omega d squared plus i omega d t plus omega 0 squared A. And this is going to be equal to f exponential i delta because I multiply both sides by exponential i delta. And this is equal to f cosine delta plus i f sine delta. Just your last equation. Any questions so far?

So look at what I have been doing. So I have this equation of motion. As usual, I go to complex notation. Then I guess Z equal to A exponential i omega d t minus delta because my friends from the math department already solved this, and I'm just following it. Then I can calculate all those terms, plug in e, and basically, you will arrive at this equation.

This equation is a complex equation. So what does that mean? That means one equation is equal to two equations because you have the real part, you have the imaginary part. Therefore, that's very nice because I have two unknowns. The first one is A, a constant. And the second one is delta. Now I have two equations I can solve what would be the functional form for A and the delta. And let me go immediately solve this equation.

So if I take the real part from this equation, basically what I'm going to get is omega 0 squared-- this is real-- minus omega d squared-- this is also real-- times A. A is actually some real number. This is actually equal to f cosine-- f0. Sorry, I missed a zero here. So that zero I missed. This should be f0. f0 cosine delta.

And I can also collect all the terms, which is imaginary terms. Then I get only the second term from the left hand side is with i in front of it. Therefore, I get omega d gamma A from the left hand side. And from the right hand side, there's only one imaginary term. Therefore, I get-- this is equal to f0 sine delta.

So now I have two equations. I have two unknowns. Therefore, I can easily solve A and delta. So I call this equation number one. I call this equation number two. So now I can-- sounding in quadrature the two equations-- in quadrature. And the left-hand side will give you A squared omega 0 squared minus omega d squared squared plus omega d squared gamma squared. That is actually coming from the second equation. That gives you the left hand side. It's a square of the sum the first and second equation.

And the right hand side will become f0 square cosine delta cosine squared delta plus sine squared delta. And this is equal to 1. So that's actually the trick to get rid of delta.

Then I can get what will be the resulting A. A is actually a function of omega d. Omega d is given to you. It's actually determined by you-- how fast do you want to oscillate this system. And this is equal to f0 divided by square root of this whole thing. So this will give you omega 0 squared minus omega d squared squared plus omega d squared gamma squared.

Then we can also calculate what would be the delta. The trick is to take a ratio between equation number two and the equation number one-- 2 divided by 1. Basically, you will get tangent delta. This is sine divided by cosine. f0 actually cancel. This is equal to what? Equal to the ratio of these two terms. After you take the ratio, A drops out. Basically, what you get is gamma omega d divided by omega 0 squared minus omega d squared.

So we have solved A and the delta through this exercise. So what does that mean? Originally, I assume my solution to be A exponential i omega d t minus delta. Therefore, I would like to go back to the real world, which is actually theta. So basically, if I take the real part, I would get theta of t, which is actually the real part of Z. And that will give you A omega d cosine omega d t minus delta is also a function of omega d.

So we have done this exercise. And you can see that the first thing which we see here is that there's no free parameter from this solution. A is decided by omega d. And delta is also decided by omega d. There's a lot of math, but actually we have overcome those and that we have a solution. But it is actually clear to you that this cannot be the full story.

Because you have a second-order differential equation, you need to have two free parameters in the solution. What is actually missing? Anybody can tell me what is missing.

AUDIENCE: The homogeneous solution.

YEN-JIE LEE: Very good. The homogeneous solution is missing. So that's actually why we actually have no free parameter here. Once is the omega d is determined, once the f0 is given, then you have the functional form which decides what is actually theta.

So what in actually the full solution? A full solution should be, as you said, a combination of homogeneous solution and the particular solution which we actually got here. So if I prepare the system to be in a situation of, for example, underdamped situation. Then what I'm going to do is actually pretty simple. What I am going to do is to just copy the underdamped solution from last lecture and combine that with my particular solution, which I obtained here.

So that actually to see what actually the full solution looks like. I have A omega d cosine omega d t minus delta is a function of omega d. This is actually so-called steady-state solution. And, of course, as you mentioned, I need to also add the homogeneous solution and basically the no -- this actually -- according to what I wrote there, I have a functional form of exponential A exponential minus gamma over 2t equals sine omega t plus alpha. So I changed A to B because I already have the A there just to avoid confusion.

Then basically, you get B exponential minus gamma over 2t cosine omega t plus alpha. Basically, they are two free parameters, B and alpha. Those two free parameters can be determined by initial conditions. So, for example, initially I actually release the rod at some fixed angle of theta initial. And also the initial velocity is 0. Then I can actually practice solution A using those initial conditions to solve B and alpha. Any questions so far? Yes.

AUDIENCE: Are we assuming it's underdamped?

YEN-JIE LEE: Yeah, I'm assuming it's underdamped, the situation. So that's the assumption. So it depends on the size of gamma and the omega 0. Then you have actually four different kinds of solution. If gamma is equal to 0, then what you are going to plug in is the solution from no damping as a your homogeneous solution. And if you prepare this system underwater, damping is colossal, it's huge, then you actually plug in the overdamped solution to be your homogeneous part of the solution. Any other questions? Very good question.

So now maybe you got confused a bit. I have now omega d. I have also omega. And there's another one we just called omega 0. What are those? So omega 0 is the natural angular frequency without given the drag force. If you remove everything just like without considering any drag force, et cetera, and that is actually the natural frequency of the system.

And what is omega? Omega, according to the function, omega is defined to be omega 0 squared minus gamma squared over 4 square root of that. That is actually the oscillation frequency, which we actually discussed last lecture, after you add drag force into it again. Finally, omega d is how fast you actually drive this system. So that is actually the definition of these three omegas.

So you can see that, if I prepare my solution to be underdamped situation, then basically you will see that this is actually so-called a steady-state solution because A omega d is a constant. So it's going to be there forever. And the second term is actually B exponential minus gamma over 2t. It's decaying as a function of time.

So if you are patient enough, you wait, then this will be gone. So that is actually how we actually understand this mathematical result. And now, of course, you can actually take a look at this. This is actually just assuming some kind of initial condition and plug in the solution and plot it as a function of time.

And you can see that this function looks really weird, looks a bit surprising. What does that mean? It looks really strange. But at some point, this superposition of these two functions-- because one of the functions actually dies out, disappears-- then you will see that, if you wait long enough, then you actually only see a very simple structure, which is oscillation frequency of omega d. And that means a large t.

In the beginning, the system will not like it. You drive it, and the system don't like it. Like if I go and shake you, in the beginning, you would not like it-- maybe. And if I shake you long enough, and you say, come on, OK, fine. I accept that. So that is actually what is going to happen to the system.

So now I would like to go through a short demonstration, which is actually the air cart, which you seen before. There's a mass and there are two springs in the front and the back of this cart. And, of course, as usual, I would turn on the air so that I make the friction smaller, but there's still some residual friction. And you will see that this mass is actually oscillating back and forth. And the amplitude can become smaller and smaller as a function of time.

Now in the right hand side, I have a motor, which actually can drive this-- I can actually shorten or increase the length of the right hand side string. Then I actually introduce a driving force by the right hand side motor. If I turn it down, this is what is going to happen.

So we can see now this motor is actually going back and forth. And it has a slightly higher frequency compared to the natural frequency. So the frequency of the motor is higher. And you can see that this cart is actually oscillating. But you can see that sometimes it pulls and sometimes it moves faster.

So you can see that it's actually moving. And it stops a little bit because they are all superposition of two different kinds of oscillating functions come into play. You can see that now. It got slowed down, and it can become faster and slower and faster. But eventually, if you wait long enough, what is going to happen? What is going to happen? If we wait long enough--

AUDIENCE: [INAUDIBLE].

YEN-JIE LEE: Exactly. So basically, if you wait long enough, as you said, you will actually just oscillate at the frequency of the driving force. You can see that this motion looks really bizarre, right? Sometimes it stops, and sometimes it actually continues to move. And are you surprised? Probably you are not surprised anymore because we know math is the language to describe nature. And indeed it predicts this kind of behavior. That's really pretty cool.

In order to help you to learn a bit how to actually translate a physical situation into mathematics, what I am going to do is to introduce you another example so that actually we can actually solve it together.

So now I would like to drive a pendulum. So I prepare a pendulum at time equal to 0. This is a string attached to a ball with mass equal to m. And the length of the string is equal to l. And the angle between the vertical direction and the direction of the string is theta. And, of course, I can actually give you initial condition X initial, which is actually measured with respect to the vertical direction, and time equal to t.

This is actually the original vertical direction, the same as this dashed line. And I can actually move the top of the string back and forth to some position. And, of course, this string is connected to the ball. And this system is actually driven from the top by the engine's hand, so the engine is actually shaking this system from the top.

And I do it really nicely. So basically, I define that the displacement, d, as a function of time, is equal to delta sine omega d t. So that is actually what I'm going to do. OK, and I would like to see what is going to happen to this pendulum.

So, as usual, the first step towards solving this problem is to define a coordinate system. So what is actually the coordinate system I'm going to use? So now I define pointing upward to be y. I define the horizontal direction pointing to the right hand side of the board to be x.

So that's not good enough. I still need the origin, right? So now I also define my origin to be the original position of the ball which is actually completely addressed before I do the experiment. So that this is actually the equilibrium position of this system actually. Then I define here to be 0, 0.

So once I have that defined, I can now express the position of this mass of this ball to be xt and yt. And see what we are going to get. Of course, as usual, we are going to analyze the force actually acting on this ball. So therefore, as usual, we will draw a force diagram.

So basically, you have the little mass here, and you have actually two forces acting on this little mass, or little ball. This is Fg pointing downward. It's a gravitational force. And now this is actually equal to minus mg y. And there's also a string tension, T. Since we have this definition of theta here, basically I have a T which is actually pointing to the upper left direction of the board. Oh, don't forget-- actually there's a third force, which is actually the F drag. F drag is actually equal to minus bx dot in the x direction.

Now I would like to write down the expression for also the string tension, T. The T is actually equal to minus T sine theta in the x direction because the T is pointing to upper left direction. So therefore, the position to x direction will be minus sine theta and plus T cosine theta in the y direction because the tension is actually pointing upper left.

As usual, this is actually pretty complicated to solve. I have this cosine. I have this sine there, right? So what I'm going to do is to assume that this angle theta is very small, as usual. So I will take small angle approximation. Then basically, you have sine theta is roughly to be equal to theta, and that is actually equal to what? Equal to-- here. Basically, you can actually calculate what will be the theta. The sine theta, or theta, would be equal to x minus d divided by l. And of course, taking a small angle approximation will bring cosine theta to be 1.

Then after this approximation, my T will become minus T x minus d divided by l in the x direction plus T in the y direction because sine theta is replaced by this approximated value because sine theta is actually replaced by 1. Any questions? Yes.

AUDIENCE: Is that a constant or is that the change in sine?

YEN-JIE LEE: It's a constant. Yeah, I was going too fast. So this is actually a constant of my amplitude.

AUDIENCE: And the drag force is only in the x direction?

YEN-JIE LEE: Yes, it's only in the x direction. So I'm only trying to actually move this point back and forth horizontally.

So now I have all the components T, Fg, f drag. And, of course, you can see that I already ignored the drag force in the y direction from that formula because I am only considering the system to be moving in the x direction. Therefore, I can now collect all the terms in the x direction. Basically, you will have m x double dot. This is equal to minus b x dot, which is actually coming from the drag force, minus T x minus d divided by l. This is actually coming from this term in the x direction.

Let's look at those in the y direction. And y double dot would be equal to minus mg plus T. The minus mg is from the gravitational force. And this T is coming from the y component of the string tension.

And of course, since we are taking a very small angle approximation, there will be no vertical motion. Yes.

AUDIENCE: Why did we use the small approximation for sine theta when we're going to use x minus d over l, which represents psi instead of just theta?

YEN-JIE LEE: Yeah, so also in this case, they happen to be exactly the same. And why I care is actually the cosine theta. Otherwise, I would have to deal with cosine theta. And also, this y double dot would not be equal to 0, which is what I'm going to assume here. Good question.

So the question was, why do I need to take an approximation? Because I want to get rid of cosine theta.

So now from this y direction, I can solve T will be equal to mg because I assume that there's no y direction motion. And I can conclude that-- originally, I don't know what is actually the string tension. It's denoted by T. Now, from this second equation, I can conclude that T will be equal to mg, which is the gravitational force.

Then, once I have that, I can go back to x direction. Basically, I get m x double dot. This is equal to minus b x dot minus mg over l x minus d. Everything is working very well. And I just have to really write down the d function explicitly. What is d? d is just a reminder, delta sine omega d t.

So I will plug that into that equation. And also I will bring all the terms related to x to the left hand side just to match my convention.

All right, so now I will be able to get the result, m x double dot plus b x dot plus mg over l x.

And that is actually equal to mg over l d. And this is equal to mg over l delta sine omega dt. So basically, I collect all the terms, put it to the left hand side. And I write down T explicitly, which is this.

Then I can divide everything by m. Then I get m x double dot plus b over m x dot plus g over l x. And that would be equal to g over l delta sine omega dt. Now, of course, as usual, I will define this to be gamma, define this to be omega 0 squared. And I would define this to be f0, which is equal to omega 0 squared delta. It happened to be equal like that. And then this actually becomes x double dot plus gamma x dot plus omega 0 squared x equal to f0 sine omega dt.

Am I going too fast? OK, everybody is following. So we see that ha! This equation-- I know that. I know this equation, right? Because we have just solved that a few minutes ago. Therefore, I know immediately what will be the solution. The solution is here already. I have A omega d and the tangent delta, the function of force there. Therefore, I can now write down what will be the A. So A is actually just equal to f0 divided by square root of omega 0 squared minus omega d squared squared plus omega d squared gamma squared.

So now the question is-- what does the result actually mean. I have this function. I have that function, tangent delta. It's solved. It's actually the amplitude of the steady-state solution and also the phase difference between the drag force phase and the steady-state oscillation phase. So that's actually the amount of lag and the size of the amplitude. But through this equation it is very difficult to understand. So what I'm going to do is to take some limit so that actually we can help you to understand what is going on.

So suppose I assume that omega d goes to 0. So what does that mean? This is the engine's hand and is moving really slowly and see what is going to happen. If I do this, then you will find that A omega d-- since omega goes to 0, this is gone, this is gone. Therefore, you will see that omega A will be equal to omega 0 divided by-- I'm sorry-- of f0 divided by omega 0 squared. And that is actually equal to g delta over l divded by g over l. And that will give you delta.

So what does that mean? This means that, if I drive this thing really slowly, then the amplitude of the mass will be equal to how much I actually move, which is delta. OK. Do you get it?

In addition to that, tangent delta-- since I am taking the limit omega d goes to 0. Therefore, tangent delta will be equal to 0, and that means delta will be equal 0. Any questions? So that means there will be no phase difference. The system has enough time to keep up with my speed.

The second limit, obviously, omega d goes to infinity. What does that mean? That means I'm going to hold this as a string and shake it like crazy really fast and see what will happen, OK?

So in that case, you will get A omega d, and that one goes to 0, because omega d goes to infinity. This one goes to 0. And also, tangent delta will go to infinity. Therefore, delta will go to pi. So that means they will be out of phase. Any questions so far in these two limits?

OK, so what I'm going to do now is to take a small toy, which I made for my son, who is one-year-old now. Because I would like him to learn wavelength vibration before he goes to quantum, right? Hey? So I made this toy. And he looked at it.

So you can see now, I can demonstrate what is going to happen when omega is approaching to 0. OK? I am already doing it. Can you see it? No? It's a very exciting experiment. Can you see that?

You see that this is the origin vertical direction. If I do it really, really, really slowly, you can see that the amplitude of the ball is actually exactly the same as the displacement I introduced. So that's kind of obvious.

So now, let's see what is going to happen if I drive this system like crazy. OK, not going up and down. Eeeee-- that's the maximum speed I can do. Maybe you can do it faster. But you can see that nothing happened. So amplitude is close to 0, because what you have been doing is-- disappear, you sort cancelling each other. And it's actually not going to contribute to the motion of this ball.

So now, you can see that I can also test, what is actually the natural frequency? And what I am going to do is to oscillate at around the natural frequency to see what is going to happen. Let's see what is going to happen.

You can see that the delta is really small, right? Can you see the delta. It's really small-- very small-- very small. But you can see that the amplitude, the A, is huge. What does that tell us? What does that tell us? Yeah?

AUDIENCE: Well, we're experiencing resonance.

YEN-JIE LEE: Yes, we are experiencing resonance. And also, that also tells you that the system is underdamped very much. The Q value is very big. So if I calculate the amplitude, A-- now, I can calculate the amplitude, A, at natural frequency. What I'm going to get is-- now, I can actually plug in omega d equal to omega 0.

So if I plug in omega d-- omega d equal to omega 0-- then what is going to happen? So this term is working. So you have A is equal to f-0 divided by omega 0 gamma. Omega-d is now equal to omega 0. And that is going to give you-- so f-0 is actually omega 0 square delta divided by omega 0 gamma.

And the one omega 0 actually cancels. Then, basically, you will get Q times delta. What is Q? Just a reminder, it's actually the ratio of omega 0 and the gamma. When the Q is very large, what does that mean? That means it's so close to an idealized situation that direct force is very small.

You can see that in the example which I have been doing. So you can see that, ah, it is really the case. So you can see that if my delta is something like 1 centimeter, but the amplitude is actually at the order of 1 meter, maybe. What does that mean?

That means that Q is actually, roughly, 100. So you can actually even get a Q out of this experiment. Any questions so far? OK, that's very good.

So now, we can go ahead and take a look at the structure of the A and the delta. As we demonstrated before, we make sense of those three different kinds of situations-- omega d goes to 0, omega d goes to infinity. And of course, I would like to know the force structure of A and delta. Therefore, what I'm going to do is to plug A omega d as a function of omega d.

So what I'm going to get is this will be equal to delta when omega d goes to 0. We just demonstrated that. And this will increase to a large value and drop down to 0, when omega d goes to infinity. And you can see that this is around omega 0.

And you are going to get a huge amplitude at around omega 0. But not quite. The maxima is actually slightly smaller than omega 0. You can actually calculate that as part of the homework. So that makes sense.

Now, I can also plug the delta, which is the phase difference-- and you can see that this phase difference will be, originally, 0, when the omega d is very small. And this is actually the omega 0. I hope you can see it. And this will be increasing rapidly here and approaching to pi.

So that means, when you are shaking this system like crazy-- very high frequency-- then the system cannot keep up with the speed. The amplitude will be very small. And also, the amplitude will be out of phase completely.

So let's actually do us another demonstration, using this little device here. This is actually what you see before, the ball with a Mexican hat. And you can see that there is a spring attached to this system. And on the top, what I am going to do is to use this motor to drive this system up and down, as a direct force.

So now, what am I going to do is to come from a very low-frequency oscillation. So you can see that the natural frequency is sort of like this. And you can see that, now, I am driving this system really slowly. You can see, this is actually going up and down really slowly. And you see that-- huh-- the amplitude is actually pretty small. There's no excitement for the moment.

All right, so what I'm going to do is, now, I increase the speed of the motor and see what will happen. So you can see, now, it's actually driving it with higher and higher frequency. You see that-- huh-- something is happening. You can see the amplitude is getting larger and larger. I'm still increasing the frequency-- increasing, increasing-- until something-- something happened! Right?

Did you see that? It starts to oscillate up and down. Because right now, you can see that-- look at the top. The frequency of the motor is now really close to the natural frequency of this system. So a resonance behavior will happen. And what you are going to get is that, OK, omega d around omega 0. Then you are going to get large time amplitude.

So now, what I am going to do is to continue to increase the driving frequency to a very large value. OK, now it's actually doing the "mmmmm"-- doing it really fast. You can see on the top-- very fast. OK, I even get it even faster. You see that-- huh-- indeed, this system is now oscillating at a larger frequency. It's trying to keep up with the driving force. But you can see that the amplitude is actually much smaller than what had happened before.

So before the class, you may actually think that, OK, drive it really fast. Maybe we'll increase the amplitude. But in reality, actually, it will give you a very small amplitude. Another thing, which is interesting to know, is that you can see that, when the driving force is actually at the maximum. And actually the position of this mass is actually at the minimum. So they are actually out of phase.

I hope you can see it. It's like this. OK, so what you can see is that, when I understand the system and I try to drive it with the natural frequency, what is going to happen is that I'm exciting this system to a state of resonance. So basically, you'll get some resonance behavior.

So I have shown you that this works for driven mechanical oscillator. It also works for the spring-mass system. And there are many other things which also work, which is around you. For example, if you happen to be my office hour, you would notice that the air-condition in my office is actually creating a resonance behavior.

You'll see low frequency sound-- "mm mm mm"-- something like that. And that is because the pipe actually happens to have the frequency match with the resonance frequen-- OK, the airflow actually happened to excite the pipe, so that it's actually oscillating up and down at that frequency.

So what I did was I tried to turn it down to low and see what happened. But unfortunately, it actually excited another resonance. I see, now, not a low-frequency sound, but a very high-frequency sound. I will post a video, actually online. So my life is hard, right?

But I'm a physicist. Is So I choose to use the median. Then I actually stay between the two resonances. Then I don't hear the additional sound, which bothers me.

Another example is that, when I was in Taiwan as a undergrad, I was living outside in a apartment. And with my flat-mate, we owned a very old washing machine. So in the middle of the night, the washing machine would started to walk around, like my flat-mate. And we are not scared.

That is because the oscillation frequency-- actually, the rotation-- happened to match with the frequency of the washing machine. Therefore, when we started to wash our clothes, it start to walk around in the room. So as a physicist, what we have decided is to make the spin slightly slower, or even faster. Then, actually, you can see that, when you do that, then you get rid of the resonance behavior. So it's not walking around any more. We can control it.

Another thing which is interesting is that the resonance behavior is not only in the physical objects, which we actually deal with these days also. But either you learn quantum mechanics and upon the field theory, you will find that there are resonance also in a mass wave function.

So basically, you can see that these are examples of the Z boson resonance peak. So if you scatter a electron and positron then, basically, you'll see that the cross-section have a resonance peak at around 90 GeV. And that is actually another very interesting example of a resonance in particle physics.

Finally, the last example, which I am going to go through is an example involving a glass. We have prepared a very high-quality glass here. Maybe you have seen this glass before. They are pretty nice. And I usually use it to enjoy my red wines, which you cannot, enjoy, probably now.

So you can see that this is the glass. And if I put a little bit of water on my hand and I rub it--

[VIBRATING TONE]

--carefully, I can actually excite one of the resonance frequencies. So you can see that we have all everything working on a single particle. And that will give you one resonance frequency. If I work on two particles, which you will see that in the next lecture, I would get two resonance frequencies. And this glass is made of infinite number of particles. Therefore, I will have infinite number of resonance frequencies.

When I'm rubbing it, I'm actually giving input of all kinds of different frequencies. But the glass will be smart enough so that it will pick up the one it likes the most, which is the resonance frequency. So you can see that the sound is actually, roughly, 683 Hertz. And you can actually measure it with your phone.

So on the TV commercial, you may have seen that there's a lady singing. And she's singing so loudly such that the glass-- "bragh!"-- breaks. Can we get a volunteer today to sing in front of us? Oh-- singing. Can you sing it-- high frequencies?

"Ahhhh."

[LAUGHTER & APPLAUSE]

Very good try. But it didn't work. OK, I guess it's really difficult to perform that in front of so many people unprepared. But fortunately, we are MIT. So we have designed a device, which actually can help us to achieve this. So this device actually contain a amplifier here. And I can now control the frequency of the sound through this scope.

And this amplifier will actually amplify the signal and produce a sound wave and try to actually isolate this glass. So we are going to do this experiment. So we will need to change the loud setting a bit. Because the sound is going to be, probably, too loud. Just for safety-- some of you may not survive.

[LAUGHTER]

So I'm handing out these. OK, who is closer? OK, maybe you.

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: Oh. Oh, sorry.

[LAUGHTER]

I'm so sorry. What? I don't need that. OK. So just for safety, I will put this on. And what I am going to do is also put these glass on. OK, maybe I'll do this first.

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: OK, so what I am going to do now is to start producing sound wave.

[LOUD TONE]

So through the camera, you should be able to see what is actually shown on the screen. So you can see that, if this glass is actually moving, the wood inside would also vibrate. So you can see that, clearly, we don't have resonance yet. So what I am going to do is to increase the frequency and see what happens.

So now, it's actually 643. It's actually still below the resonance frequency. Now, I have measured the frequency. It should be 684. So now, it's actually 653-- 663 Hertz-- 673 Hertz. Can you see the movement? You cannot see the movement yet.

683-- you see? You see, now--

AUDIENCE: Yes.

YEN-JIE LEE: --the frequency of the sound is actually matching with one of the natural frequencies of the glass. Apparently, the glass likes it. And now, you can see that it is still vibrating. And the next step, which we are going to do, is to try to increase the amplitude, increase the volume of the sound, and see what happens.

Maybe you want to cover your ears, just for safety.

[LOUD TONE]

OK, then the glass may break, if we are lucky. Let us see what is going to happen.

[INCREASINGLY LOUDER TONE]

Oh!

[APPLAUSE]

TECH SUPPORT: Good job.

YEN-JIE LEE: Very good.

TECH SUPPORT: Perfect. That's the quickest one we've had.

YEN-JIE LEE: Yeah, thank you very much. Thank you, glass.

[LAUGHTER]

So you can see the power of resonance. So if I tune down the frequency slightly more, then you will be where? You'll be here. Then you will not have enough amplitude to break the glass. And also, as we discussed before, the quality of the glass should be really, really high, such that the resulting amplitude will be very large. Then you can actually break it with a external sound wave.

And if we go above the resonance frequency, then you would not also move a bit. Because if you go to very large omega d, then amplitude will be pretty small. OK, let me try to switch back to my presentation. I think we did. Sure.

So this is actually what we have learned today. We have learned the behavior of a damped driven oscillator. We have learned the transient behavior. So what is actually transient behavior is a mixture of steady state solution, which was coming from the driving force, and the homogeneous solution. If you wait long enough, this will decay and disappear. Is

And we have learned resonance. So an IOC circuit, which you actually solved that in your Pset, in pendulum, which I just show you, which helped my son to learn wavelength vibrations-- and air condition, washing machine, glass-- particle physics. We can see damped os-- driven oscillator or resonance almost everywhere.

So I hope that you enjoyed the lecture today. And what we are going to do next time is to put multiple objects together so that you see the interaction between one particle to the other particle and see how we can actually make sense of this kind of system. Thank you very much. And I will be here if you have additional questions related to the lecture.