Wave Packet Statistics

Here we'll answer the question "What are the position and momentum of my particle, and how well do I know them?".

The wave packet amplitude function $\Psi(x)$

We start with the same wave function we had last week; a wave packet centered at x_0 with width d and wave-number k_0.

In[1]:=
```
$Assumptions = {d > 0, ℏ > 0, Element[{x0, k0, d, ℏ}, Reals]}
```
Out[1]=
```
{d > 0, ℏ > 0, (x0 | k0 | d | ℏ) ∈ Reals}
```

In[2]:=
```
Ψ = A Exp[-(x - x0)^2 / (2 d^2)] Exp[I k0 (x - x0)] /. A → 1/(Sqrt[d] π^(1/4))
```
Out[2]=
$$\frac{e^{i\,k0\,(x-x0)-\frac{(x-x0)^2}{2\,d^2}}}{\sqrt{d}\,\pi^{1/4}}$$

In[3]:=
```
Ψ̃ = 1 / Sqrt[2 π] Integrate[Ψ Exp[-I k x], {x, -Infinity, Infinity}]
```
Out[3]=
$$\frac{\sqrt{d}\,e^{-\frac{1}{2}d^2\,(k-k0)^2-i\,k\,x0}}{\pi^{1/4}}$$

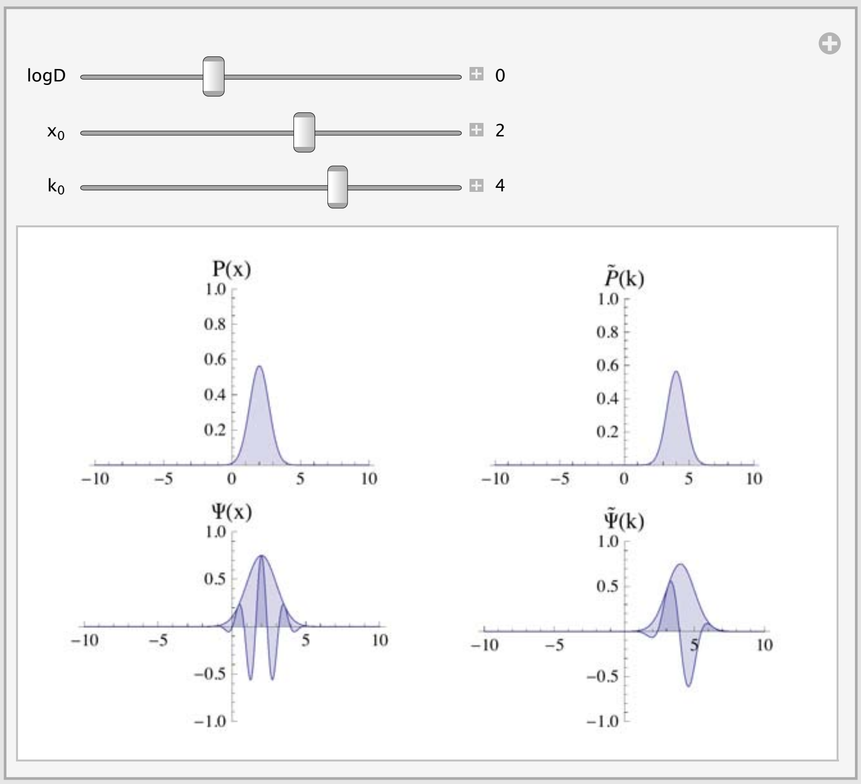
The probability distribution functions (PDFs) for $\Psi(x)$ and $\tilde{\Psi}(k)$

The PDFs of x and k are shown below.

In[4]:=

```
Px = Abs[Ψ]^2;
Pk = Abs[Ψ̃]^2;
Manipulate[GraphicsGrid[
  {{Plot[Px /. {d → 10^logD, x0 → x₀, k0 → k₀}, {x, -10, 10}, PlotRange → {0, 1},
      Filling → Axis, ImageSize → Small, PlotLabel → "P(x)"],
     Plot[Pk /. {d → 10^logD, x0 → x₀, k0 → k₀}, {k, -10, 10}, PlotRange → {0, 1},
      Filling → Axis, ImageSize → Small, PlotLabel → "P̃(k)"]},
    {Plot[{Re[Ψ], Abs[Ψ]} /. {d → 10^logD, x0 → x₀, k0 → k₀}, {x, -10, 10},
      PlotRange → {-1, 1}, Filling → Axis, ImageSize → Small, PlotLabel → "Ψ(x)",
      PerformanceGoal → Quality], Plot[{Re[Ψ̃], Abs[Ψ̃]} /. {d → 10^logD, x0 → x₀, k0 → k₀},
      {k, -10, 10}, PlotRange → {-1, 1}, Filling → Axis, ImageSize → Small,
      PlotLabel → "Ψ̃(k)", PerformanceGoal → Quality]}}],
 {{logD, 0}, -1, 2, 0.1, Appearance → "Labeled"},
 {{x₀, 2}, -10, 10, 0.5, Appearance → "Labeled"},
 {{k₀, 4}, -10, 10, 0.5, Appearance → "Labeled"}]
```

Out[6]=

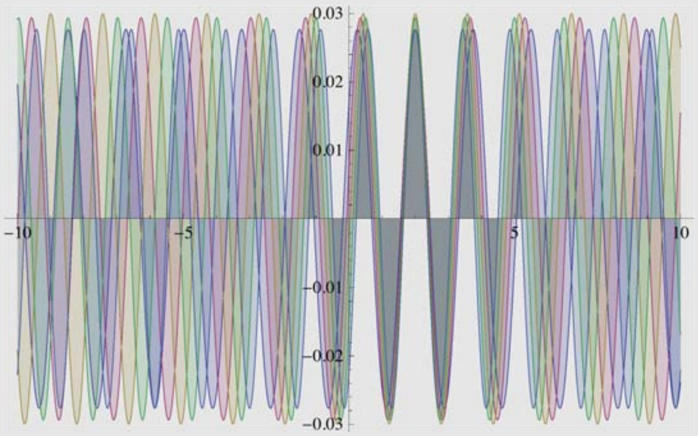


Why the width in $\tilde{\Psi}(k)$ given only k_0 in $\Psi(x)$?

Let’s try to make the real part of our wave packet from a finite collection of sine-waves.

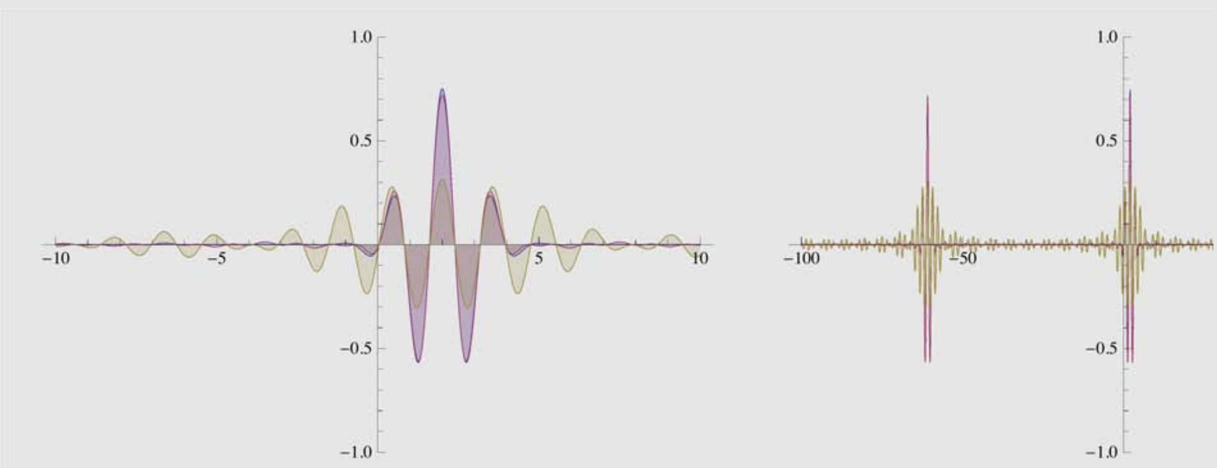
In[7]:=
```
values = {d → 1, x0 → 2, k0 → 4};
dk = d / 10;
PsiWave = Re[Ψ̃ Exp[I k x]] dk / Sqrt[2 π] /. {k → (k0 + dk #)} /. values &;
waves = {PsiWave[-4], PsiWave[-2], PsiWave[0], PsiWave[2], PsiWave[4]};
Plot[waves, {x, -10, 10}, Filling → Axis, PerformanceGoal → Quality]
```

Out[11]=



In[12]:=
```
SumPsiWave = Sum[PsiWave[n], {n, -#, #}] &;
waves = {Re[Ψ] /. values, SumPsiWave[20], SumPsiWave[5]};
GraphicsRow[
 {Plot[waves, {x, -10, 10}, PlotRange → {-1, 1}, Filling → Axis, ImageSize → Medium],
  Plot[waves, {x, -100, 100}, PlotRange → {-1, 1}, Filling → Axis]}]
```

Out[14]=



The expectation value and uncertainty in x and k

Just what you would expect.

In[15]:=
```
x̄ = Integrate[x Px, {x, -Infinity, Infinity}]
```

Out[15]=
```
x0
```

In[16]:=
```
k̄ = Integrate[k Pk, {k, -Infinity, Infinity}]
```
Out[16]=
```
k0
```

Though you might be a bit uncertain.

In[17]:=
```
Δx = Simplify[Sqrt[Integrate[(x - x̄)^2 Px, {x, -Infinity, Infinity}]]]
```
Out[17]=
$$\frac{d}{\sqrt{2}}$$

In[18]:=
```
Δk = Simplify[Sqrt[Integrate[(k - k̄)^2 Pk, {k, -Infinity, Infinity}]]]
```
Out[18]=
$$\frac{1}{\sqrt{2}\, d}$$

Putting this all together we see that $\Delta x = d/\sqrt{2}$ and $\Delta k = 1/\sqrt{2}\, d$ such that $\Delta x\, \Delta k = 1/2$, which de Broglie tells us means $\Delta x\, \Delta p = \hbar/2$.

The momentum operator $\hat{p}$

Let's try out this new trick, the momentum operator $\hat{p} = -i\, \hbar\, \partial_x$.

In[19]:=
```
pPsi = Simplify[-I ℏ ∂x Ψ]
```
Out[19]=
$$\frac{e^{\frac{(x-x0)\,(2\, i\, d^2\, k0 - x + x0)}{2\, d^2}} \left(d^2\, k0 + i\, (x - x0)\right) \hbar}{d^{5/2}\, \pi^{1/4}}$$

Is the wave-packet a momentum eigenstate?

In[20]:=
```
Simplify[pPsi / Ψ]
```
Out[20]=
$$\frac{\left(d^2\, k0 + i\, (x - x0)\right) \hbar}{d^2}$$

No, since $\hat{p}\, \Psi = \hbar\, \left(k_0 + i\, (x - x0) \big/ d^2\right)\, \Psi$, so the prefactor is not a constant (it is a function of x). But not to worry, we can still compute <p> and Δp.

In[21]:=
```
p̄ = Integrate[Ψ* (-I ℏ ∂x Ψ), {x, -Infinity, Infinity}]
```
Out[21]=
```
k0 ℏ
```

In[22]:=
```
p2 = Integrate[Ψ* (-I ℏ ∂x (-I ℏ ∂x Ψ)), {x, -Infinity, Infinity}]
```

Out[22]=
$$\frac{1}{2}\left(\frac{1}{d^2}+2\,k0^2\right)\hbar^2$$

In[23]:=
```
Δp = Simplify[Sqrt[p2 - p^2]]
```

Out[23]=
$$\frac{\hbar}{\sqrt{2}\,d}$$

Which is what we expect, since $p = \hbar k$.