Delta functions and complex exponentials

The magic of delta functions

Let's look at how delta functions work.

In[1]:=
```
$Assumptions = {A > 0, x0 > 0, D > 0, Element[{A, x0, D}, Reals]}
```
Out[1]=
```
{A > 0, x0 > 0, D > 0, (A | x0 | D) ∈ Reals}
```

We start with an unnormalized wave-function involving a delta function at x_0.

In[2]:=
```
Ψu = A DiracDelta[x - x0]
```
Out[2]=
```
A DiracDelta[x - x0]
```

So, let's try to normalize it.

In[3]:=
```
Ainv = Integrate[Abs[Ψu]^2 , { x, -Infinity, Infinity}]
```
Out[3]=

$$\int_{-\infty}^{\infty} \text{Abs}[\text{A DiracDelta}[x - x0]]^2 \, dx$$

That's *Mathematica*'s way of saying "I can't integrate that". You can imagine why; the delta function is defined so as to pick out the value of a function at $x = x_0$, but in this case the value it is "picking out" is the ∞ of the other delta function.

In[4]:=

$$\int_{-\infty}^{\infty} \text{DiracDelta}[x - x1] \, \text{DiracDelta}[x - x2] \, dx$$

Out[4]=
```
DiracDelta[-x1 + x2]
```

We can't integrate δ^2, so we can't normalize this wave-function. That sounds bad, but let's go on and leave A in place. We can take the Fourier transform to find $\tilde{\Psi}_u$ anyway.

In[5]:=
```
Ψ̃u = 1 / Sqrt[2 π] FullSimplify[Integrate[Ψu Exp[-I k x], { x, -Infinity, Infinity}]]
```
Out[5]=

$$\frac{A\, e^{-i\, k\, x0}}{\sqrt{2\, \pi}}$$

We get a complex exponential (a definite momentum state). However...

In[6]:=
```
Ainv = Simplify[Integrate[Abs[Ψ̃u]^2 , { k, -Infinity, Infinity}]]
```
Out[6]=
```
∞
```

... we can't normalize this one either. Can we transform back to position space?

In[7]:=

```
Ψ2 = FullSimplify[1 / Sqrt[2 π] Integrate[Ψ̃u Exp[I k x], {k, -Infinity, Infinity}]]
```

Integrate::idiv : Integral of $e^{i\,k\,(x-x0)}$ does not converge on $\{-\infty, \infty\}$. ≫

Out[7]=

$$\frac{\int_{-\infty}^{\infty} \frac{A\, e^{i\,k\,(x-x0)}}{\sqrt{2\,\pi}}\, dk}{\sqrt{2\,\pi}}$$

Not easily, but we can observe that the integral is infinite only when $x = x_0$ and doesn't amount to much away from this point, which sounds a lot like a delta function. Adding the fact that the oscillations become very fast as D→∞, such that the integral over any dx away from x_0 will go to zero by averaging over these oscillations, the identification with a delta function becomes stronger.

In[8]:=

```
Ψ2 = FullSimplify[1 / Sqrt[2 π] Integrate[Ψ̃u Exp[I k x], {k, -D, D}]]
```

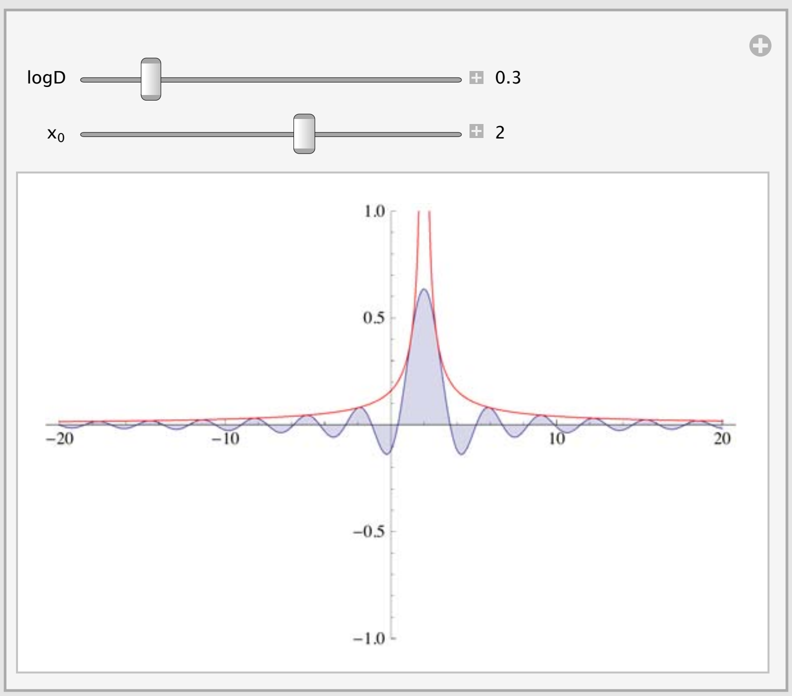
Out[8]=

$$\frac{A\,\mathrm{Sin}[D\,(x - x0)]}{\pi\,x - \pi\,x0}$$

In[9]:=

```
Manipulate[Show[Plot[Ψ2 /. {A → 1, x0 → x0, D → 10^logD}, {x, -20, 20},
    PlotRange → {-1, 1}, Filling → Axis, PerformanceGoal → Quality],
   Plot[Abs[1 / (π x - π x0)] /. {x0 → x0}, {x, -20, 20}, PlotRange → {-1, 1},
    PlotStyle → Red, Filling → None, PerformanceGoal → Quality]],
  {{logD, 0.3}, 0, 2, 0.1, Appearance → "Labeled"},
  {{x0, 2}, -10, 10, 0.1, Appearance → "Labeled"}]
```

Out[9]=



We can also note that if we assert $\int_{-\infty}^{\infty} e^{i\,k\,(x-x0)}\, dk = 2\,\pi\,\delta(x - x_0)$, we will have returned to Ψ_u (without ever properly normalizing our wave functions). Alternately, we can show this by introducing a Gaussian envelope to help us with the integral, which we will later remove by taking the limit as D→∞ (making the envelope infinitely wide).

In[10]:=

```
del = FullSimplify[∫_{-∞}^{∞} e^{i k (x-x0) - (k/D)^2} dk] / (2 π)
```

Out[10]=

$$\frac{D\, e^{-\frac{1}{4} D^2 (x-x0)^2}}{2\sqrt{\pi}}$$

This is the now familiar transform of a Gaussian wave-packet, which will become infinitely narrow and tall as D→∞. Using this like a delta function in an integral with some other smooth function...

In[11]:=

```
delInt = Integrate[del (1 - x^2), {x, -Infinity, Infinity}]
```

Out[11]=

$$1 - \frac{2}{D^2} - x0^2$$

In[12]:=

```
delIntLim = Limit[delInt, D → Infinity]
```

Out[12]=

$$1 - x0^2$$

... we can see that it works as expected.

8.04 Quantum Physics I
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