A Chopped Wave

Remember that sine-wave between D and -D that I tried to present? I thought that only I could make such a mistake and it was better that we never speak of it again, but then Prof. Adams showed the same thing in lecture! I guess we should do it out right...

The wave function Ψ(x)

We start by defining our piecewise function,

In[1]:=
```
$Assumptions = {n > 0, k0 > 0, ℏ > 0, Element[{A, k0, ℏ}, Reals], Element[n, Integers]}
```
Out[1]=
```
{n > 0, k0 > 0, ℏ > 0, (A | k0 | ℏ) ∈ Reals, n ∈ Integers}
```

In[2]:=
```
Ψu = Piecewise[{{A Sin[k0 x], x > -D && x < D}}] /. D → n π / k0
```
Out[2]=

$$\begin{cases} A\,\mathrm{Sin}[k0\,x] & x > -\frac{n\pi}{k0}\ \&\&\ x < \frac{n\pi}{k0} \\ 0 & \mathrm{True} \end{cases}$$

and normalizing it. Notice that this time I was careful to ensure continuity with D→n π/k0, with n ∈ Integers.

In[3]:=
```
Anorm = FullSimplify[Solve[Integrate[Abs[Ψu]^2 , { x, -Infinity, Infinity}] == 1, A]]
Ψ = Ψu /. Anorm[[2]][[1]];
```
Out[3]=

$$\left\{\left\{A \to -\frac{\sqrt{\frac{k0}{n}}}{\sqrt{\pi}}\right\}, \left\{A \to \frac{\sqrt{\frac{k0}{n}}}{\sqrt{\pi}}\right\}\right\}$$

which simplifies to A=1/D.

With that done, we can take the Fourier transform to find $\tilde{\Psi}$.

In[5]:=
```
Ψ̃ = 1 / Sqrt[2 π] FullSimplify[Integrate[Ψ Exp[-I k x], { x, -Infinity, Infinity}]]
```
Out[5]=

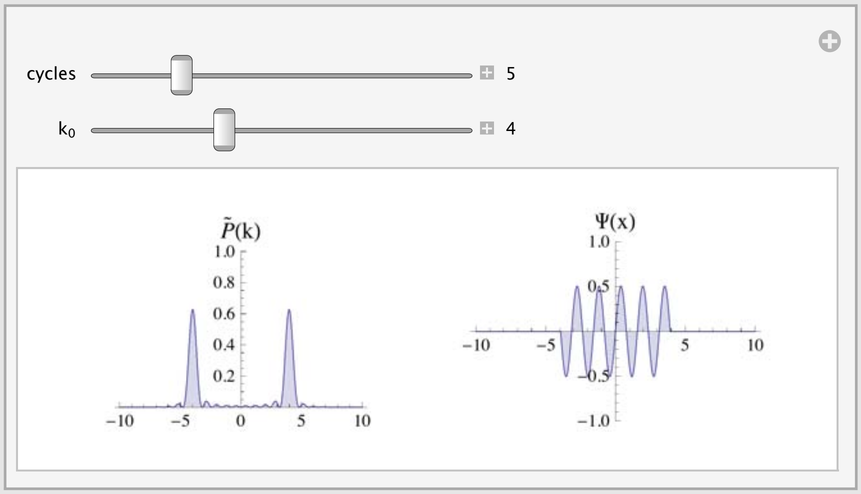
$$\frac{i\,(-1)^n \sqrt{2}\ k0 \sqrt{\frac{k0}{n}}\ \mathrm{Sin}\left[\frac{k\,n\,\pi}{k0}\right]}{\left(-k^2 + k0^2\right)\pi}$$

And to get a feel for this, let's plot $\tilde{P}$ along with Ψ.

In[6]:=

```
Px = Abs[Ψ]^2;
Pk = Abs[Ψ̃]^2;
Manipulate[GraphicsRow[{Plot[Pk /. {n → cycles, k0 → k₀}, {k, -10, 10}, PlotRange → {0, 1},
     Filling → Axis, ImageSize → Small, PerformanceGoal → Quality, PlotLabel → "P̃(k)"],
   Plot[Ψ /. {n → cycles, k0 → k₀}, {x, -10, 10}, PlotRange → {-1, 1},
     Filling → Axis, PerformanceGoal → Quality, PlotLabel → "Ψ(x)"]}],
 {{cycles, 5}, 1, 20, 1, Appearance → "Labeled"},
 {{k₀, 4}, 1, 10, 0.1, Appearance → "Labeled"}]
```

Out[8]=



Here we see that this is a tricky one. As was pointed out to me in recitation last week, the momentum distribution has peaks at positive and negative values around k_0. In fact, it may be surprising to some just how concentrated around $\pm k_0$ the PDF of k turns out to be, especially when many cycles are involved.

Let's compute some statistics on this wave function. It is clear from symmetry that <x> = 0 and <k> = 0, but let's write the integrals anway.

In[9]:=

```
x̄ = Integrate[x Px, {x, -Infinity, Infinity}]
k̄ = Integrate[k Pk, {k, -Infinity, Infinity}]
```

Out[9]= 0

Out[10]= 0

It is less clear what the uncertainties are in x and k. Let's start with Δx...

In[11]:=

```
Δx = Simplify[Sqrt[Integrate[(x - x̄)^2 Px, {x, -Infinity, Infinity}]]]
```

Out[11]=

$$\frac{\sqrt{-3 + 2\, n^2 \pi^2}}{\sqrt{6}\ k0}$$

which for large n goes to $\frac{D}{\sqrt{3}}$. Fine. Onto Δk...

In[12]:=
```
Δk = Sqrt[Integrate[(k - k̄)^2 Pk, {k, -Infinity, Infinity}]]
```

Integrate::idiv : Integral of $k^2\,\mathrm{Abs}\left[\frac{\mathrm{Sin}\left[\frac{k\,n\,\pi}{k0}\right]}{-k^2+k0^2}\right]^2$ does not converge on $\{-\infty, \infty\}$. ≫

Out[12]=
$$\sqrt{\int_{-\infty}^{\infty}\frac{2\,e^{-2\,\pi\,\mathrm{Im}[n]}\,k^2\,\mathrm{Abs}\left[\frac{k0\sqrt{\frac{k0}{n}}\,\mathrm{Sin}\left[\frac{k\,n\,\pi}{k0}\right]}{-k^2+k0^2}\right]^2}{\pi^2}\,dk}$$

This may have just failed because I don't know how to convince *Mathematica* that n is really an integer. Let's be explicit...

In[13]:=
```
Δk = Simplify[Sqrt[Integrate[(k - k̄)^2 Pk /. n → 2, {k, -Infinity, Infinity}]]]
Δk = Simplify[Sqrt[Integrate[(k - k̄)^2 Pk /. n → 10, {k, -Infinity, Infinity}]]]
```

Out[13]= k0

Out[14]= k0

so Δk is probably k_0 for any integer value of n. In this case we find Δx Δk > 1/2 for all integer values of n.

In[15]:=
```
Simplify[{Δx Δk, Δx Δk /. n → 1, N[Δx Δk /. n → 1]}]
```

Out[15]=
$$\left\{\frac{\sqrt{-3+2\,n^2\,\pi^2}}{\sqrt{6}},\ \sqrt{\frac{1}{6}\left(-3+2\,\pi^2\right)},\ 1.67029\right\}$$

The momentum operator $\hat{p}$

Let's try out this new trick, the momentum operator $\hat{p} = -i\,\hbar\,\partial_x$.

In[16]:=
```
pPsi = -I ℏ ∂x Ψ
```

Out[16]=
$$-i\,\hbar\left(\begin{cases}\frac{k0\sqrt{\frac{k0}{n}}\,\mathrm{Cos}[k0\,x]}{\sqrt{\pi}} & \frac{n\,\pi}{k0}+x>0\ \&\&\ -\frac{n\,\pi}{k0}+x<0\\ 0 & \mathrm{True}\end{cases}\right)$$

In[17]:=
```
p̄ = Integrate[Ψ* (-I ℏ ∂x Ψ), {x, -Infinity, Infinity}]
```

Out[17]= 0

In[18]:=

```
p2 = Integrate[Ψ* (-I ℏ ∂x (-I ℏ ∂x Ψ)), {x, -Infinity, Infinity}]
```

Out[18]=

$$\frac{k0^2\,\hbar^2\,(2\,n\,\pi - \mathrm{Sin}[2\,n\,\pi])}{2\,n\,\pi}$$

In[19]:=

```
Δp = Simplify[Sqrt[p2 - p^2]]
```

Out[19]=

$k0\,\hbar$

Which is what we expect, since $p = \hbar k$, and this worked without specifying a value of n.