Quantum Physics I (8.04) Spring 2016
Assignment 1

Massachusetts Institute of Technology
Physics Department
February 4, 2016

Due R, February 11, 2016
5:00pm

Announcements

- Please put you name and section number at the top of your problem set, and place it in the 8.05 box labeled with your section number near 8-395 by 5:00pm.
- You may find it fun to read the first few pages of Dirac's book on Quantum Mechanics.

Problem Set 1

1. **Radiative collapse of a classical atom.** [10 points]

   In a classical universe, we might try to build a hydrogen atom by placing an electron in a circular orbit around a proton. We know, however, that a non-relativistic, accelerating electron radiates energy at a rate given by the Larmor formula:

   $$\frac{dE}{dt} = -\frac{2}{3}\,\frac{e^2 a^2}{c^3}\,.$$

   Here e is the electron charge and a is the magnitude of the electron acceleration. So the classical atom may have a stability problem. We want to figure out how big is this effect. In the units we are working the electron potential energy in the presence of the proton is $V = -e^2/r$ and the magnitude of the force of attraction is e^2/r^2.

   (a) Show that for a non-relativistic electron the energy ΔE lost per revolution is small compared to the electron's kinetic energy K. Do this by computing the ratio $\Delta E/K$. Hence, it is possible to regard the orbit as circular at any instant, even though the electron eventually spirals into the proton.

   (b) A good estimate for the size of the hydrogen atom is 50 pm (pico-meters), and a good estimate for the size of the nucleus is 1fm (femto-meter). Compare the classically calculated velocity of the electron to the velocity of light at an orbital radius of 50 pm, 1 pm, and 1 fm.

   (c) Calculate how long it would take for the electron to spiral from 50 pm to 1pm? Are you justified in ignoring relativistic corrections? Would the answer using the non-relativistic approximation change much for a spiral from 50pm to 1fm?

   (d) As the electron approaches the proton, what happens to its energy? Is there a minimum value of the energy the electron can have?

2. **Quantized energies.** [5 points]

   Consider an electron in circular motion around a fixed (heavy) proton as a model for the hydrogen atom. Let $V = -e^2/r$ denote the potential energy of the electron.

   (a) Assume a circular orbit and find the relations between the kinetic energy K of the electron, its potential energy V and the total energy E.

   (b) Assume that the magnitude L of the electron angular momentum is quantized and equal to $n\hbar$ where n is a positive integer. Find the quantized values E_n for the total energy and the associated orbit radii r_n. Express your answers in terms of n, the rest energy $E_e = m_e c^2$ of the electron, its Comption wavelength $\lambda\!\!\!^{-} = \frac{\hbar}{m_e c}$, and the fine structure constant $\alpha = \frac{e^2}{\hbar c}$.

3. **DeBroglie Relations and the Scale of Quantum Effects.** [10 points]

   (a) <u>Matter Particles as Waves</u>

   If a wavelength can be associated with every moving particle, then why are we not forcibly made aware of this property in our everyday experience? In answering, calculate the de Broglie wavelength $\lambda = h/p$ (with $h = 6.6 \times 10^{-34}$J.s) of each of the following particles:

   i. an automobile of mass 2000 kg traveling at a speed of 50 mph (22m/s)
   ii. a marble of mass 10 g moving with a speed of 10 cm/s,
   iii. a smoke particle of diameter 100nm and a mass of 1fg being jostled about by air molecules at room temperature ($T = 300$K) (assume that the particle has the same translational kinetic energy as the thermal average of the air molecules, $KE = \frac{3}{2}k_B T$, with $k_B = 1.38 \times 10^{-23}$J/K)
   iv. an ^{87}Rb atom that has been laser cooled to a temperature of $T = 100\mu K$. Again, assume $KE = \frac{3}{2}k_B T$.

   (b) <u>Light Waves as Particles</u>

   The Photoelectric effect suggests that light of frequency ν can be regarded as consisting of photons of energy $E = h\nu$, where $h = 6.6 \times 10^{-34} Js$.

   i. Visible light has a wavelength in the range of 400-700 nm. What are the energy and frequency of a photon of visible light?
   ii. The microwave in my kitchen operates at roughly 2.5 GHz at a max power of 300W. How many photons per second can it emit? What about a low-power laser (10mW at 633 nm), or a cell phone (0.25W at 850MHz)?
   iii. How many such microwave photons does it take to warm a 200ml glass of water by 10 C? (The heat capacity of water is roughly 4J/gK, and the density is 1g/ml.)
   iv. At a given power of an electromagnetic wave, do you expect a classical wave description to work better for radio frequencies, or for X-rays?

4. **Complex Number Practice** [15 points]

A complex number can be written in either Cartesian or polar form

$$z = a + ib = re^{i\theta}\,, \quad |z| \equiv \sqrt{a^2 + b^2}\,. \tag{1}$$

The real numbers a and b are, respectively, the real and imaginary parts of z. The real numbers r and θ are, respectively, the magnitude and phase of z. We call $|z|$ the norm of z. Use this definition for z in the following:

(a) Use Taylor expansions to derive the Euler formula

$$e^{i\theta} \;=\; \cos\theta + i\sin\theta\,. \tag{2}$$

(b) Write a and b in terms of r and θ, and vice versa.

(c) Complex numbers are viewed a vectors in a two-dimensional "complex plane". Multiplication of a complex number by a phase (a complex number of unit magnitude) is equivalent to *a rotation* in the complex plane.

i. Show that multiplication by i is equivalent to rotation by $90°$: $iz = re^{i(\theta+\pi/2)}$

ii. Write iz in terms of a and b. What is the real part of iz?

iii. Show that multiplication by $e^{i\phi}$ is equivalent to rotating by ϕ.

(d) The complex conjugate z^* of a complex number $z = a + ib$ is $z^* = a - ib$. A complex number z is actually real if $z = z^*$, meaning that its imaginary part is zero. A complex number z is actually imaginary if $z = -z^*$, which implies that its real part is zero.

i. Is there a number that is both real and purely imaginary?

ii. What is $(z^*)^*$? Show that $z^* = re^{-i\theta}$.

iii. Express the real and imaginary parts of z in terms of z and z^*.

iv. Show that zz^* is real and evaluate it to express it in terms of a and b, in terms of r, and in terms of $|z|$.

(e) Using the Euler formula derive formulae for $\cos 2\theta, \sin 2\theta, \cos 3\theta$, and $\sin 3\theta$, all in terms of $\sin\theta$ and $\cos\theta$. Derive formulae for $\cos(A+B)$ and $\sin(A+B)$, both in terms of sines and cosines of A and B.

5. **Absorption?** [5 points]

A photon collides with a free electron. Explain why the photon cannot be completely absorbed.

6. **Mach-Zender interferometer** [10 points]

Consider the Mach-Zender interferometer and assume an input beam of the form $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Call P_0 and P_1 the detection probabilities at D0 and D1.

(a) Calculate P_0 and P_1 assuming we insert a phase shifter with phase δ_l on the *lower* leg of the interferometer.

(b) Calculate P_0 and P_1 assuming we insert a phase shifter with phase δ_u on the *upper* leg of the interferometer.

(c) Calculate P_0 and P_1 assuming we insert the two phase shifters simultaneously.

7. **Elitzur-Vaidman bombs!** [10 points]

(a) Suppose you decide to test bombs with a Mach-Zender interferometer repeatedly until the status of any given bomb is certain beyond reasonable doubt. What fraction of the working bombs are certified without detonation?

(b) Suppose 80% of the bombs in your possession are defective. You choose one at random and test it with a Mach-Zender interferometer by sending in one photon. You detect the photon at D0. What is the probability that the bomb is defective?