PARTICLE NATURE OF LIGHT AND WAVE NATURE OF MATTER

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1 Photoelectric Effect

The photoelectric effect was first observed by Heinrich Hertz in 1887. When polished metal plates are irradiated, he observed, they may emit electrons, then called "photo-electrons". The emitted electrons thus produce a *photoelectric current*. The key observations were:

- There is a threshold frequency ν_0. Only for frequencies $\nu > \nu_0$ is there a photoelectric current. The frequency ν_0 depends on the metal and the configuration of the atoms at the surface. It is also affected by inhomogeneities.
- The magnitude of the photoelectric current is proportional to the intensity of the light source.
- Energy of the photoelectrons is *independent* of the intensity of the light source.

A natural explanation for the features in this effect didn't come until 1905, when Einstein explained the above features by postulating that the energy in light is carried by discrete quanta (later called photons) with energy $h\nu$. Here h is Planck's constant, the constant used by Planck to to produce a fit for the blackbody energy as a function of frequency.

Figure 1: Electrons in a metal are bound. If the photon energy is greater than the work function W an electron may be ejected.

A given material has a characteristic energy W, called the *work function*, which is the minimum energy required to eject an electron. This is not easily calculated because it is the result of an interaction of many electrons with the background of atoms. It is easily measured, however. When the surface of the material is irradiated, electrons in the material absorb the energy of the incoming photons. If the energy imparted on an electron by the absorption of a single photon is greater than the work function W, then the electron is ejected with kinetic energy E_{e^-} equal to the difference of the photon energy and the work function:

$$E_{e^-} \;=\; \tfrac{1}{2}mv^2 \;=\; h\nu - W \;=\; E_\gamma - W. \tag{1.1}$$

This equation, written by Einstein explains the experimental features noted above, once we assume that the quanta act on individual electrons to eject them. The threshold frequency is defined by

$$h\nu_0 = W\,, \tag{1.2}$$

as it leads to a photoelectron with zero energy. For $\nu > \nu_0$ the electrons will be ejected. Increasing the intensity of the light source increases the rate that photons arrive, which will increase the magnitude of the current, but will not change the energy of the photoelectrons because it does not change the energy of each incoming quanta.

Equation (1.2) allowed Einstein to make a prediction: The kinetic energy of the photo-electrons increases linearly with the frequency of light. Einstein's prediction was confirmed experimentally by Millikan (1915) who measured carefully the photoelectron energies and confirmed their linear dependence on the energy. Millikan's careful work allowed him to determine the value of Planck's constant $\hbar$ to better than 1% accuracy! Still, skepticism remained and physicists were not yet convinced about the particle nature of these light quanta.

Example: Consider UV light with wavelength $\lambda = 290$nm incident on a metal with work function $W = 4.05$eV What is the energy of the photo-electron and what is its speed?

Solution: It is useful to solve these problems without having to look up constants. For this try recalling this useful relation

$$\hbar c \;=\; 197.33 \text{ MeV.fm}\,, \qquad \hbar \equiv \frac{h}{2\pi}\,, \tag{1.3}$$

where MeV $= 10^6$eV and fm$= 10^{-15}$m. Let us use this to compute the photon energy. In this case,

$$E_\gamma = h\nu = 2\pi\hbar\frac{c}{\lambda} \;=\; \frac{2\pi\cdot 197.33\text{ MeV.fm}}{290\times 10^{-9}\text{m.}} \;=\; \frac{2\pi\cdot 197.33}{290}\text{ eV} \;\approx\; 4.28\,\text{eV}, \tag{1.4}$$

and thus

$$E_{e^-} = E_\gamma - W = 0.23\,\text{eV}. \tag{1.5}$$

To compute the energy we set

$$0.23\,\text{eV} \;=\; \tfrac{1}{2}m_e v^2 \;=\; \tfrac{1}{2}(m_e c^2)\big(\tfrac{v}{c}\big)^2 \tag{1.6}$$

Recalling that $m_e c^2 \simeq 511{,}000$eV one finds

$$\tfrac{0.46}{511000} \;=\; \big(\tfrac{v}{c}\big)^2 \quad\to\quad \tfrac{v}{c} \;=\; 0.0009488\,. \tag{1.7}$$

With and $c = 300,000$ Km/s we finally get $v \simeq 284.4\,$Km/s.

This is a good point to consider units, in particular the units of h. We can ask: Is there a physical quantity that has the units of h. The answer is yes, as we will see now. From the equation $E = h\nu$, we have

$$[h] = \left[\frac{E}{\nu}\right] = \frac{ML^2/T^2}{1/T} = L\cdot M\frac{L}{T}\,, \tag{1.8}$$

where $[\cdot]$ gives the units of a quantity, and M, L, T are units of mass, length, and time, respectively. We have written the right-most expression as a product of units of length and momentum. Therefore

$$[h] \ = \ [\mathbf{r} \times \mathbf{p}] = [\mathbf{L}]. \tag{1.9}$$

We see that h has units of angular momentum! Indeed for a spin one-half particle, the magnitude of the spin angular momentum is $\frac{1}{2}\hbar$.

With $[h] = [r][p]$ we also see that one has a canonical way to associate a length to any particle of a given mass m. Indeed, using the speed of light, we can construct the momentum $p = mc$, and then the length ℓ is obtained from the ratio h/p. This actually is the **Compton wavelength** ℓ_C of a particle:

$$\lambda_C = \frac{h}{mc} \tag{1.10}$$

then has units of length; this is called the *Compton wavelength* of a particle of mass m. Note that this length is independent of the velocity of the particle. The de Broglie wavelength of the particle uses the true momentum of the particle, not mc! Thus, Compton and de Broglie wavelengths should not be confused!

It is possible to get some physical intuition for the Compton wavelength λ_C of a particle. We claim that λ_C *is the wavelength of a photon whose energy is equal to the rest energy of the particle.* Indeed we would have

$$mc^2 \ = \ h\nu \ = \ h\,\tfrac{c}{\lambda} \quad \rightarrow \quad \lambda = \frac{h}{mc}\,, \tag{1.11}$$

confirming the claim. Suppose you are trying to localize a point particle of mass m. If you use light, the possible accuracy in the position of the particle is roughly the wavelength of the light. Once we use light with $\lambda < \lambda_C$ the photons carry more energy than the rest energy of the particle. It is possible then that the energy of the photons go into creating more particles of mass m, making it difficult, if not impossible to localize the particle. The Compton wavelength is the length scale at which we need *relativistic quantum field theory* to take into account the possible processes of particle creation and annihilation.

Let us calculate the Compton wavelength of the electron:

$$\lambda_C(e) = \frac{h}{m_e c} \ = \ \frac{2\pi\hbar c}{m_e c^2} \ = \ \frac{2\pi \cdot 197.33\,\text{MeV.fm}}{0.511\,\text{MeV}} \ = \ 2426\,\text{fm} \ = \ 2.426\,\text{pm}. \tag{1.12}$$

This length is about 20 times smaller than the Bohr radius (53 pm.) and about two-thousand times the size of a proton (1 fm.). The Compton wavelength of the electron appears in the formula for the change of photon wavelength in the process called Compton scattering.

2 Compton Scattering

Originally Einstein did not make clear that the light quantum meant a particle of light. In 1916, however, he posited that the quantum would carry momentum as well as energy, making the case for a particle much clearer. In relativity, the energy, momentum, and rest mass of a particle are related by

$$E^2 - p^2c^2 \ = \ m^2c^4. \tag{2.13}$$

(Compare this with the classical equation $E = p^2/2m$.) Of course, one can also express the energy and momentum of the particle in terms of the velocity:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.14}$$

You should use these expressions to confirm that (2.13) holds ($|\mathbf{p}| = p$). A particle that moves with the speed of light, like the photon, must have zero rest mass, otherwise its energy and momentum would be infinite due to the vanishing denominators. With the rest mass set to zero, equation (2.13) gives the relation between the photon energy E_γ and the photon momentum p_γ:

$$E_\gamma = p_\gamma c. \tag{2.15}$$

Then, using $\lambda\nu = c$, we reach

$$p_\gamma = \frac{E_\gamma}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}. \tag{2.16}$$

We will see this relation again later when we discuss matter waves.

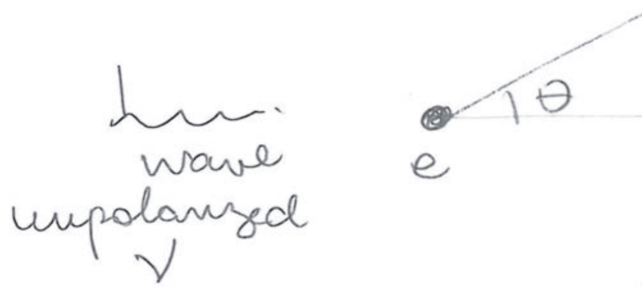

Figure 2: Unpolarized light incident on an electron scatters into an angle θ. Classically, this is described by Thomson scattering. The light does not change frequency during this process.

Compton carried out experiments (1923–1924) scattering X-rays off a carbon target. X-rays correspond to photon energies in the range from 100 eV to 100 KeV. The goal was scattering X-ray photons off free electrons, and with some qualification, the electrons in the atoms behave this way.

The classical counterpart of the Compton experiment is the scattering of electromagnetic waves off free electrons, called *Thompson scattering*. Here an electromagnetic wave is incident on a electron. The electric field of the wave shakes the electron which oscillates with the frequency of the incoming field. The electron oscillation produces a radiated field, of the same frequency as that of the incoming radiation. In classical Thomson scattering the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \tfrac{1}{2}\left(1+\cos^2\theta\right), \tag{2.17}$$

where θ is the angle between the incident and scattered wave, with the radiated energy at the same frequency as the incoming light. This is shown in Figure 2. The cross-section has units of length-squared, or area, as it should. It represents the area that would extract from the incoming plane wave the amount of energy that is scattered by the electron. Indeed the quantity $e^2/(mc^2)$ is called the classical electron radius and it is about 2.8 fm! not much bigger than a proton!

If we treat the light as photons, the elementary process going on is a collision between two particles; an incoming photon and a roughly stationary electron. Two facts can be quickly demonstrated:

- The photon cannot be absorbed by the electron. It is inconsistent with energy and momentum conservation (exercise)

- The photon must lose some energy and thus the final photon wavelength λ_f must be larger than the initial photon wavelength λ_i. This is clear in the laboratory frame, where the initially stationary electron must recoil and thus acquire some kinetic energy.

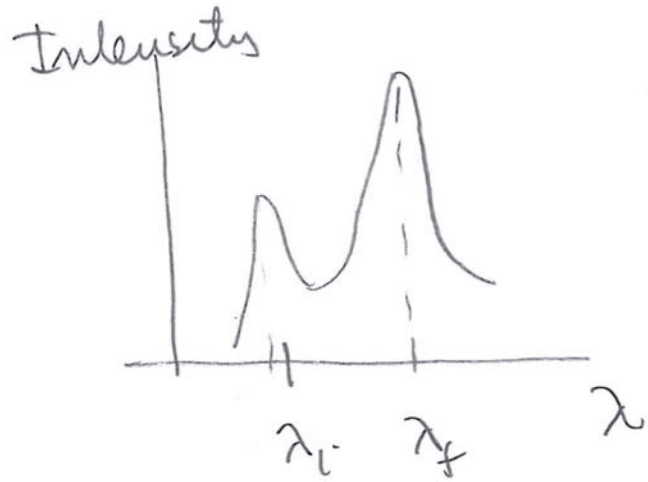

Figure 3: The results of Compton's scattering experiment. The incident photon wavelength is λ_i, and the scattered photon wavelength is $\lambda_f \simeq \lambda_i + \ell_C$, corresponding to $\theta = 90°$.

Indeed, Compton's observations did not agree with the predictions of Thompson scattering: the X-rays changed frequency after scattering. A calculation using energy and momentum conservation shows that the change of wavelength is correlated with the angle between the scattered photon and the original photon:

$$\lambda_f = \lambda_i + \frac{h}{m_e c}(1 - \cos\theta) \;=\; \lambda_i + \ell_C\,(1 - \cos\theta)\,. \tag{2.18}$$

Note that appearance of the Compton wavelength of the electron, the particle the photon scatters off from. The maximum energy loss for the photon occurs at $\theta = \pi$, where

$$\lambda_f(\theta = 180°) = \lambda_i + 2\lambda_C\,. \tag{2.19}$$

The maximum possible change in wavelength is $2\lambda_C$. For $\theta = \frac{\pi}{2}$ the change of wavelength is exactly ℓ_C

$$\lambda_f(\theta = 90°) = \lambda_i + \lambda_C\,. \tag{2.20}$$

Compton's experiment used molybdenum X-rays with energy and wavelength

$$E_\gamma \approx 17.5\,\text{keV}\,, \qquad \lambda_i = 0.0709\,\text{nm}\,, \tag{2.21}$$

incident on a carbon target. Placing the detector at an angle $\theta = 90°$ the plot of the intensity (or number of photons scattered) as a function of wavelength is shown in Figure 2. One finds a peak for for $\lambda_f = 0.0731$ nm, but also a second peak at the original wavelength $\lambda_i = 0.0709$ nm.

The peak at λ_f is the expected one: $\lambda_f - \lambda_i \simeq 2.2\,\text{pm}$, which is about the Compton wavelength of 2.4 pm. Given that the photons have energies of 17 KeV and the bound state energies of carbon are about 300 eV, the expected peak represents instances where the atom is ionized by the collision and it is a fine approximation to consider the ejected electrons. The peak at λ_i represents a process in which an electron receives some momentum from the photon but still remains bound. This is not very unlikely: the typical momentum of a bound electron is actually comparable to the momentum of the photon. In this case the photon scatters at 90° and the recoil momentum is carried by the whole atom. The relevant Compton wavelength is therefore that of the atom. Since the mass of the carbon atom is several thousands of times larger than the mass of the electron, the Compton wavelength of the atom is much smaller than the electron Compton wavelength and there should be no detectable change in the wavelength of the photon.[1]

3 Matter Waves

As we have seen, light behaves as both a particle and a wave. This kind of behavior is usually said to be a **duality:** the complete reality of the object is captured using *both* the wave and particle features of the object. The photon is a particle of energy E_γ, but has frequency ν which is a wave attribute, with $E = h\nu$. It is a particle with momentum p_γ but it also has a wavelength λ, a wave attribute, given by (2.16)

$$\lambda = \frac{h}{p_\gamma}. \tag{3.22}$$

In 1924, Louis de Broglie proposed that the wave/particle duality of the photon was universal, and thus valid for matter particles too. In this way he conjectured the *wave nature of matter*. Inspired by (3.22) de Broglie postulated that associated to a matter particle with momentum p there is a plane wave of wavelength λ given by

$$\lambda = \frac{h}{p}. \tag{3.23}$$

This is a fully quantum property: if $h \to 0$, then $\lambda \to 0$, and the particles have no wave properties. And exciting consequence of this is that matter particles can diffract or interfere! In the famous Davisson-Germer experiment (1927) electrons are strike a metal surface and one finds that at certain angles there are peaks in the intensity of the scattered electrons. The peaks showed the effect of constructive interference from scattering off the lattice of atoms in the metal, demonstrating the wave nature of the electrons. One can also do two-slit interference with electrons, and the experiment can be done shooting one electron at a time. A recent experiment [arXiv:1310.8343] by Eibenberger *et.al* reports interference using molecules with 810 atoms and mass exceeding 10 000 amu (that's 20 million times the mass of the electron!)

The de Broglie wavelength can be calculated to estimate if quantum effects are important. Consider for this purpose a particle of mass m and momentum p incident upon an object of size x, as illustrated in Figure 3. Let $\lambda = h/p$ denote the de Broglie wavelength of the particle. The wave nature of the particle is not important if λ is much smaller than x. Thus, the "classical approximation," in which wave effects are negligible, requires

$$\text{Wave effects negligible:} \qquad \frac{\lambda}{x} \ll 1. \tag{3.24}$$

Using $\lambda = h/p$, this yields

$$\text{Wave effects negligible:} \qquad x\,p \gg h\,, \tag{3.25}$$

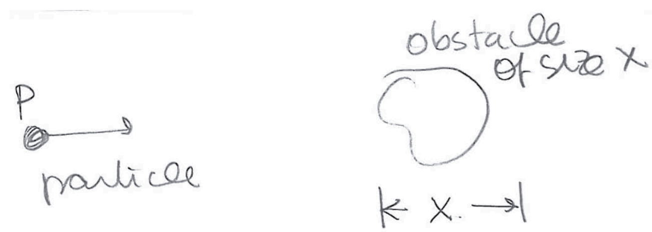

Figure 4: A particle of momentum p incident on an obstacle of size x.

a relation in which both sides have units of angular momentum.

Classical behavior is a subtle limit of quantum mechanics: a classical electromagnetic field requires a large number of photons. Any state with an exact, fixed number of photons, even if large, is not classical, however. Classical electromagnetic states are so-called coherent states, in which the number of photons fluctuates.

Andrew Turner transcribed Zwiebach's handwritten notes to create the first LaTeX version of this document.

[1]Thanks to V. Vuletic for a clarification of this point.