That's a solution. It's an accomplishment to have such a solution. If somebody gives you a value of the energy, you can calculate what is the phase shift, but we probably want to do more with it.

So you decide to plot this on a computer. Again, there's lots of variables going on here, so you would want to figure out what are the right variables to plot this.

And the right variables suggest themselves. From k squared equal 2 me over h squared, unit less constant are things like ka, k prime a, and that's it.

Well, so ka is a proxy for the energies. OK, a squared is really 2me, a squared over h bar squared. And so this we could call anything.

Well, let's call it u. On the other hand, k prime squared then-- if you have k prime a squared that it's also unit free would be 2me a squared over h squared plus 2mv0 a squared over h squared.

You probably recognize them. The first one is just u squared. I should call this u squared, sorry. U squared, and this is our friend z0 squared. It's that number that tells you the main thing you always want to know about a square well.

That ratio between the energy v0 to the demand to the energy that you can build with h bar m and a. So here we go. We have k prime a given by this quantity, and therefore let me manipulate this equation.

Might as well do it. It probably easier to consider just tan delta, which is the inverse of this. You would have 1 minus the inverse of this would be k prime a over ka, put the a's always, so cot k prime a tan ka over tan ka plus k prime aka cot k prime a.

So in terms of our variables, see k prime a is the square root of this, so k prime a square root of u squared plus z0 squared, and k prime a over ka, you divide now by u. So it's square root of 1 plus z0 squared over u squared. That's this quantity. So how big, how much space do I need to write it? Probably, I should write it here.

1 minus square root of 1 plus z0 squared over u squared cot k prime a is the square root of z0 squared plus u squared and tan of k a, which is u over tan u plus square root of 1 plus z0 over u squared cotangent of square root of z0 squared plus u squared. OK, it's not terrible. That's tan delta.

So if somebody gives you a potential, you calculate what z0 is for this potential, you put z0 there, and you plot as a function of u with Mathematica. And plotting as a function of u is plotting as a function of ka. And that's perfectly nice thing to do. And it can be done with this expression.

In this expression, you can also see what goes on when u goes to 0. Not immediately, it takes a little bit of thinking, but look at it. As u goes to 0, well, these numbers are 1, that's perfectly OK. That seems to diverge, goes like 1 over u, but u going to 0. This goes to 0. So the product goes to a number.

So the whole-- the numerator goes to a number, some finite number. On the other hand, when u goes to 0, the denominator will go to infinity, because while this term goes to 0 the tan u, this number is finite. And here you have a 1/u. So the denominator goes to infinity. And the numerator remains finite. So as u goes to 0, tangent of delta goes to zero.

So you can choose delta to be 0 for 0 energy. So as u goes to 0, you get finite divided by infinity, and goes to zero. So tan delta goes to 0. And we can take delta of ka equals 0, which is u to be 0. The phase shift is 0 for 0 energy.

Let me go here. So here is an example. z0 squared equal 3.4. That actually correspond to 0.59pi for z0. z0 equal 0.59pi. You may wonder why we do that, but let me tell you in a second.

So here are a couple of plots that occur. So here is u equals ka. And here's the phase shift, delta of u. You have the tangent of delta, but the phase shift can be calculated. And what you find is that, yes, it starts at 0, as we mentioned. And then it starts going down, but it stabilizes at minus pi, which is a neat number. That's what the phase shift does.

The so-called scattering amplitude, well you could say, when is this scattering strongest? When you get an extra wave of this propagating more strongly? So you must plot sine squared delta and sine squared is highest for minus pi over 2. So this goes like this, up, and decays as a function of u.

Third thing, the delay, is 1/a. The delay is 1/a d delta dk, as a function of u. So that, you can imagine, that takes a bit of time, because you would have to find the derivative of delta with respect to u, and do all kinds of operations. Don't worry, you will have a bit of exercises on this to do it yourselves.

But here the delay turns out to be negative. And this is unit-free. And here, comes to be equal minus 4 for equals 0, and goes down to 0.

So in this case, the delay is negative. So the reflected packet comes earlier than you would expected, which is possible, because the reflected packet is going slowly here. Finally, at this point, reaches more kinetic energy, just-- and then back.

So that's the delay. And you can plot another thing. Actually it's kind of interesting, is the quantity a, this coefficient here. That gives you an idea of how big the wave function is in the well. How much does it stick near the well?

So it peaks to 1. And it actually goes like this, and that's the behavior of this form. Basically, it does those things. So, so far so good. We got some information.

And then you do a little experiment, and try, for example, z0 equals 5. And you have delta as a function of u, and here is minus pi, minus 2pi. And actually, you find that it just goes down, and approaches now minus 2pi.

So actually, if you increase this z0 a bit, it still goes to pi, a pi excursion of the phase. But suddenly, at some value, it jumps. And it now goes to 2pi. And if you do with a larger value, at some point it goes to 3pi and 4pi. And it goes on like that.

Well if z0 would have been smaller, like half of this, the phase would go down and would go back up, wouldn't go to minus pi. It does funny things. So what's really happening is that there is a relation between how much the phase moves, and how many bound states this potential has.

And you say, why in the world? This calculation had nothing to do with bound states. Why would the phase shift know about the bound states? Well actually, it does. And here is the thing. If you remember, you've actually solved this problem in homework, the half square well, in which you put an infinite wall here.

And if you had the full square well, from minus a to a, this problem has all the old solutions of the full square well. All the old solutions exist. And if you remember the plots that you would do in order to find solutions, you have pi/2, pi, 3pi/2, 2pi. And here is the even solution. Here is the odd solution. I'll do it like that. Here is an even solution. Here is an odd solution.

And I marked the odd solutions, because we care about the odd ones, because that's what this potential has. So z0 equals 0.59pi is a little more than pi/2. So it corresponds to one solution. So there is one bound state for this z0. z0 equals 5 is about here. it's in between 3pi/2 and this. And there's two nodes, two intersections. Therefore, two solutions in the square well. And here we have that the phase has an excursion of, not just pi for one, but 2pi.

And if you did this experiment for awhile, you would convince yourself there's a magic relation between how much the phase shift moves, and how many bound states you have in this potential. This relation is called Levinson's theorem. And that's what we're going to prove in the last half an hour of this lecture.