BARTON ZWIEBACH: The next thing I want to talk about for a few minutes is about the node theorem. Theorem. And it's something we've seen before. We've heard that if you have a one-dimensional potential and you have bound states, the ground state has no nodes. The first excited state has 1 node. Second, 2, 3, 4.

All I want to do is give you a little intuition as to why this happens. So this will be an argument that is not mathematically very rigorous, but it's fairly intuitive and it captures the physics of the problem. So it begins by making two observations.

So in the node theorem, if you have psi 1, psi 2, psi 3, all energy-- energy-- eigenstates of a one-dimensional potential-- bound states. Bound states. With energy E1 less than E2, less than E3 and E4, psi n has n minus 1 nodes. Those are points where the wave function vanishes inside the range of x.

So for this square well, you've proven this by calculating all the energy eigenstates. The first state is the ground state. It has no nodes. The next state is the first excited state. It has one node. And you can write all of them, and we saw that each one has one more node than the next.

Now I want to argue that in an arbitrary potential that has bound states, this is also true. So why would that be true for an arbitrary potential? The argument we're going to make is based on continuity. Suppose you have a potential like this-- V of x-- and I want to argue that this potential will have bound states and will have no node, 1 node, 2 nodes, 3 nodes. How could I argue that?

Well, I would do the following. Here is the argument. Identify the minimum here. Call this x equals 0.

Oh, I want to say one more thing and remind you of another fact that I'm going to use. So this is the first thing, that the square well realizes this theorem, and the second is that psi of x0 being equal to psi prime at x0 being equal to 0 is not possible. The wave function and its derivative cannot vanish at the same point.

Please see the notes about this. There is an explanation in last lecture's notes. It is fact that for a second order differential equation, psi and psi prime tell you how to start the solution, and if both psi and psi prime are equal to 0, the general solution of the differential equation is always 0 everywhere.

So this kind of thing doesn't happen to a wave function-- the point where it's 0 and the derivative is 0. That never happens. This happens-- 0 wave function with the derivative. But this, no. Never happens. So those two facts.

And now let's do the following. Let's invent a new potential. Not this potential, but a new one that I'll mark the point minus a here and the point a here and invent a new potential that is infinite here, infinite there, and has this part I'll write there.

So this will be called the screened potential. Screened potential. Va of x in which Va of x is equal to V of x for x less than a, and it's infinity for x greater than a. So that's a potential in which you turn your potential into an infinite square well whose bottom follows the potential. It's not flat.

And now, we intuitively argue that as I take a to infinity, the bound states of the screened potentials become the bound states of your original potential. Because when the screen is very, very, very far away, up to infinity, you've got all your potential, and by the time you have bound states that are decaying, so the screen is not going to do much at infinity. And anyway, you can move it even further away. If you move it one light year away or two light years away, shouldn't matter.

So the idea is that the bound states-- bound states-- of Va of x as a goes to infinity are the bound states of V of x. And moreover, as you slowly increase the width of the screen, the bound states evolve, but they evolve continuously. At no point a bound state blows up and reappears or does something like that. It just goes continuously.

These are physically reasonable, but a mathematician would demand a better explanation. But that's OK. We'll stick to this. So let's continue there.

So here is the idea, simply stated. If a is going to 0, if the width of the screen is extremely narrow, you're sitting at the bottom of the potential at x equals 0. And the screened potential is basically a very, very narrow thing, and here, there's the bottom of the potential. And for sufficiently small a-- since you picked the bottom of the potential there-- it's basically flat.

And then I can use the states of the infinite square well potential. As a goes to 0, yes, you have a ground state with no nodes, a first excited state with one node, and all the states have the right number of nodes because they are the states of the infinite square well, however narrow it is.

So the only thing we have to now show is that if you have a wave function-- say, let's begin with one with no nodes-- as you increase the width of the screen, you cannot get more nodes. It's impossible to change the number of nodes continuously. So here it is. I'm going to do a little diagram.

So for example, let's assume the screen is this big at this moment, that you have some ground state like this. You've been growing this, and then as the screen grows bigger, you somehow have maybe a node. Could this have happened? As you increase this screen, you get a node.

Now I made it on this point. I didn't intend to do that, so let me do it again somewhere. Do you get a node?

Well, here was the original screen, and here the derivative psi prime is negative. Psi prime is negative. On the other hand, psi prime here is already positive.

So as you grew this screen, this PSI prime that was here must have turned from negative to positive, the way it looks here. But for that, there must have been a point somewhere here when it was horizontal if it's continuous. And therefore, there must have been some point at which psi and psi prime were both 0 at the endpoint x equals a, whatever the value of a was, because psi prime here is positive, and here is negative.

So at some point it was 0, but since it's at the point where you have the infinite square well, psi is also 0. And you would have both psi and psi prime equal 0, which is impossible. So basically, you can't quite flip this and produce a node because you would have to flip here, and you can't do it.

One could try to make a very precise, rigorous argument, but if you have another possibility that you might think, well, you have this wave function maybe. And then suddenly it starts doing this, and at some stage, it's going to try to do this. But before it does that, at some point, it will have to be just like this and cross, but at this point, psi and psi prime would be 0.

So you can intuitively convince yourself that this thing doesn't allow you to produce a node. So if you start with whatever wave function that has no nodes, as you increase the screen, you just can't produce a node. So the ground state of the whole big potential will have no nodes.

And if you start with the first excited state that has one node, as you increase the screen, you still keep one node. So the next state of the full potential will have one node as well. And that way, you argue that all your bound states of the complete potential will just have the right number of nodes, which is 0, 1, 2, 3, 4. And it all came, essentially, from the infinite square well and continued.