

Quantum Physics II (8.05) Fall 2013

Assignment 3

Massachusetts Institute of Technology
Physics Department
September 20, 2013

Due Friday, September 27, 2013
3:00pm

Problem Set 3

1. Spin one-half states along an arbitrary axis. [10 points]

- (a) An (unnormalized) spin state is given by

$$(1 + i)|+\rangle - (1 + i\sqrt{3})|-\rangle.$$

What direction does this spin state point to?

- (b) Consider the following sequence of experiments:

- i. First, prepare a beam of spin-1/2 atoms which are all in the state $|+\rangle$ by passing a beam through a Stern-Gerlach device oriented in the \hat{z} direction, and keeping only those atoms measured to have eigenvalue $+\hbar/2$.
- ii. Then, pass these atoms through a second Stern-Gerlach device designed to measure $\hat{S}_{\vec{n}}$, for some \vec{n} in the (x, z) -plane. That is, $\phi = 0$; $\theta \neq 0$. Keep only those atoms which have eigenvalue $\hbar/2$.
- iii. Finally, pass the atoms which remain through a third Stern-Gerlach experiment, oriented in the same (z) direction as the first.

Of all the atoms that entered the second magnet, what fraction are found by the third magnet to be in the state $|+\rangle$? What fraction are found by the third magnet to be in the state $|-\rangle$? What fraction never made it to the third magnet? Your answers will of course be functions of the angle θ . Argue that your answers make sense for $\theta = 0$ (\vec{n} in the z direction), $\theta = \pi/2$ (\vec{n} in the x direction) and $\theta = \pi$ (\vec{n} in the $-z$ direction).

2. Overlap of two spin one-half states. [10 points]

Consider a spin state $|\mathbf{n}; +\rangle$ where \mathbf{n} is the unit vector defined by the polar and azimuthal angles θ and ϕ and the spin state $|\mathbf{n}'; +\rangle$ where \mathbf{n}' is the unit vector defined by the polar and azimuthal angles θ' and ϕ' . Let γ denote the angle between the vectors \mathbf{n} and \mathbf{n}' :

$$\mathbf{n} \cdot \mathbf{n}' = \cos \gamma.$$

Show by direct computation that the overlap of the associated spin states is controlled by *half* the angle between the unit vectors:

$$|\langle \mathbf{n}'; + | \mathbf{n}; + \rangle|^2 = \cos^2 \frac{\gamma}{2}.$$

3. **Rotation of spin states.** [10 points]

We define the operator $\hat{R}_{\mathbf{n}}(\alpha)$, with α real and \mathbf{n} a unit vector, by

$$\hat{R}_{\mathbf{n}}(\alpha) \equiv \exp\left(-\frac{i\alpha\hat{S}_{\mathbf{n}}}{\hbar}\right) = \exp\left(-i\frac{\alpha}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right),$$

where we noted that $\hat{S}_{\mathbf{n}} = \frac{\hbar}{2}\mathbf{n}\cdot\boldsymbol{\sigma}$.

- (a) Using the definition of the exponential function and properties of the σ -matrices show that

$$\hat{R}_{\mathbf{n}}(\alpha) = I \cos \frac{\alpha}{2} - i \boldsymbol{\sigma} \cdot \mathbf{n} \sin \frac{\alpha}{2}.$$

Verify by direct computation that $\hat{R}_{\mathbf{n}}(\alpha)$ is unitary.

- (b) For brevity we write $\hat{R}_y(\alpha)$ for $\hat{R}_{\hat{e}_y}(\alpha)$ Evaluate the operator

$$\hat{R}_y(\alpha)\hat{S}_z\hat{R}_y(\alpha)^\dagger$$

in terms of \hat{S}_x , \hat{S}_y , and \hat{S}_z .

- (c) Find the state obtained by acting with $\hat{R}_y(\alpha)$ on $|+\rangle$. For what operator is the resulting state an eigenstate with eigenvalue $\hbar/2$. Explain why we can think of $\hat{R}_y(\alpha)$ as a rotation operator. (Similarly, one can show that $\hat{R}_{\mathbf{n}}(\alpha)$ is a rotation operator around an axis pointing along \mathbf{n} .)

4. **Schwarz inequality and triangle inequality** [15 points]

- (a) For real vector spaces the familiar dot product satisfies the Schwarz inequality

$$(\vec{a} \cdot \vec{b})^2 \leq (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}), \quad \text{or} \quad |\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|. \quad (1)$$

Note that in the last inequality, the vertical bars on the left-hand side denote absolute value, but on the right-hand side they denote length of the vector. Prove this inequality as follows. Consider the vector $\vec{a} - \lambda\vec{b}$, with λ a real constant. Note that

$$f(\lambda) \equiv (\vec{a} - \lambda\vec{b}) \cdot (\vec{a} - \lambda\vec{b}) \geq 0,$$

for all λ and therefore the minimum over λ is still non-negative

$$\min_{\lambda} f(\lambda) \geq 0.$$

When is the Schwarz inequality saturated?

- (b) For a complex vector space the Schwarz inequality reads

$$|\langle a|b \rangle|^2 \leq \langle a|a \rangle \langle b|b \rangle, \quad \text{or} \quad |\langle a|b \rangle| \leq |a| |b|. \quad (2)$$

Here the norm is defined by $|a|^2 = \langle a|a \rangle$. Prove this inequality using the vector $|v(\lambda)\rangle \equiv |a\rangle - \lambda|b\rangle$, with λ a *complex* constant and noting that

$$f(\lambda) \equiv \langle v(\lambda)|v(\lambda)\rangle \geq 0$$

for all λ so that its minimum over λ is non-negative.* When is the Schwarz inequality saturated?

- (c) For a complex vector space one has the triangle inequality

$$|a + b| \leq |a| + |b|, \quad (3)$$

where the norm is defined by $|a|^2 = \langle a|a \rangle$. Prove this inequality starting from the expansion of $|a + b|^2$. You will have to use the property $|\operatorname{Re}(z)| \leq |z|$, which holds for any complex number z , as well as the Schwarz inequality. Show that the equality in (3) holds if and only if $a = cb$ for c a *real* positive constant.

5. Exercises in linear algebra. [10 points]

- (a) (from Axler's book) Consider the following statement: U_1, U_2 , and W are subspaces of V and the following holds

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W.$$

Can you conclude that $U_1 = U_2$? Namely, are they the same subspace? If yes, prove it. If no, give a counterexample.

- (b) Prove that \mathbb{F}^∞ (as defined in equation (1.4)) is an infinite dimensional vector space. (Comment: There may be several ways of showing this. I found it useful to use the result (stated in the notes) that in a finite dimensional vector space the length of any spanning list must be larger than or equal to the length of any list of linearly independent vectors.)
- (c) Show that T is injective if and only if $\operatorname{null} T = \{0\}$.

To minimize over a complex variable (such as λ) one must vary the real and imaginary parts. Equivalently you can show that you can treat λ and λ^ as if they were independent variables in the sense of partial derivatives (this may be good to discuss in recitation!)

6. Basis independent quantities. [10 points]

Consider a vector space V and a change of basis from (v_1, \dots, v_n) to (u_1, \dots, u_n) defined by the linear operator $A : v_k \rightarrow u_k$, for $k = 1, \dots, n$. The operator is clearly *invertible* because, letting $B : u_k \rightarrow v_k$, we have $BA : v_k \rightarrow v_k$, showing that $BA = I$ and $AB : u_k \rightarrow u_k$, showing that $AB = I$. Thus B is the inverse of A .

- (a) Consider the mapping equations

$$u_k = A v_k, \quad \text{and} \quad v_k = B u_k,$$

and write them explicitly using the matrix representation of A in the v -basis and the matrix representation of B in the u -basis. Show that these two matrices are inverses of each other.

Consider now the linear operator T in V . Let $T_{ij}(\{v\})$ denote its matrix representation in the v basis and $T_{ij}(\{u\})$ denote its matrix representation in the u basis.

- (b) Find a matrix relation between $T_{ij}(\{v\})$ and $T_{ij}(\{u\})$, written in terms of the matrix representative of A and its inverse.
- (c) Show that the trace of the matrix representation of T is basis independent.
- (d) Show that the determinant of the matrix representation of T is basis independent.

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