Quantum Physics III (8.06) — Spring 2018

Assignment 6

Posted: Monday, April 2, 2018

Readings and Announcements

- See in Materials the PDF and LaTeX files for the proposals you will have to submit.
- Griffiths: Sections 9.2 and 9.3 for interactions of atoms with light. Chapter 10 for the adiabatic approximation.

1. **Decay of the Three Dimensional Harmonic Oscillator (15 points)**

   The object of this problem is to calculate the lifetime of a particle with charge q and mass m in the first excited states of a three-dimensional isotropic harmonic oscillator of frequency ω.

   By analogy with the hydrogen atom, we refer to the states $|1,0,0\rangle$, $|0,1,0\rangle$, $|0,0,1\rangle$ as the $2p$ states, and we call the ground state $|0,0,0\rangle$ the $1s$ state. An alternate basis for the $2p$ states is given by eigenstates of L_z.

$$
\begin{aligned}
|m_\ell = 1\rangle &= \frac{|1,0,0\rangle + i|0,1,0\rangle}{\sqrt{2}} \\
|m_\ell = 0\rangle &= |0,0,1\rangle \\
|m_\ell = -1\rangle &= \frac{|1,0,0\rangle - i|0,1,0\rangle}{\sqrt{2}}
\end{aligned}
$$

   (a) Calculate the transition rate $\Gamma(2p, m_\ell \to 1s)$ per unit time for the particle to spontaneously emit electromagnetic radiation and make a transition to the ground state. Show that the transition rate is independent of m_ℓ and give your formula for $\Gamma(2p \to 1s)$ in terms of m, ω, q, and fundamental constants.

   (b) What is the lifetime of the $2p$ state? Thinking of this as a model of hydrogen, let the particle be an electron and set $\hbar\omega = \frac{3}{4}E_{Ry}$ to give the lifetime in seconds. (E_{Ry} = Rydberg = 13.6eV)

2. **1D model of ionization (15 points)**

Consider an electron in the *ground state* of a deep one-dimensional square well:

$$V(x) = \begin{cases} 0 & \text{for } \; x < 0 \\ -V_0 & \text{for } \; 0 < x < a, \quad V_0 > 0 \\ 0 & \text{for } \; x > 0. \end{cases}$$

A very deep well means

$$V_0 \gg \frac{\hbar^2}{ma^2} \quad \rightarrow \quad \frac{2ma^2 V_0}{\hbar^2} \equiv z_0^2 \gg 1\,.$$

An electromagnetic plane wave with electric field $E(t) = 2E_0 \cos(\omega t)$ parallel to the x axis acts on the electron. The electron can then escape the well in an "ionization" process.

(a) Find the relevant density of final states in the continuum. Use momentum eigenstates *unmodified* by the well. What is the condition on ω for this to be a reasonable approximation?

(b) Calculate the transition rate from the ground state to the continuum of momentum states. You will do the following approximations:
– Use the *infinite* square-well ground state wavefunction as the initial state.
– Assume the energy of the electron on the ground state is $-V_0$.

3. **Comparing rates for spontaneous and stimulated emission (10 points)**

For downward transitions with energy difference $\hbar\omega_0$ consider the unit-free ratio r formed by dividing the spontaneous emission rate by the stimulated emission rate, where blackbody radiation at a temperature T is the stimulus:

$$r \;\equiv\; \frac{\text{spontaneous emission rate}}{\text{stimulated emission rate}}$$

(a) What is the ratio r as a function of ω_0 and T?

(b) Consider a single mode of frequency ω_0 of the electromagnetic field, associated to a photon of fixed polarization and fixed direction of propagation. Calculate (using statistical physics) the expected number $\bar{n}$ of such photons in the radiation at temperature T. Express r in terms of $\bar{n}$ and interpret the result.

(c) At room temperature, what is the frequency ν_0 (in Hz) for which both rates are the same? Which process dominates for frequencies associated with visible light? Which process dominates at the frequency 10^{10}Hz used in masers?

4. **Griffiths 9.11, p.359. (20 points) Decays of $2S, 2P$ states of hydrogen.**

5. **Griffiths 9.14, p.363. (20 points) Decays of $3S$ state of hydrogen.**