PROFESSOR: Let's do adiabatic evolution really now. Evolution. We're going to say lots of things, but the take away message is going to be the following. We're going to get maybe even confused as we do this, but the take away message is the following. You sort of begin in some quantum state, and you're going to remain in that quantum state as it changes.

All the states are going to be changing. The quantum states are going to be changing in time. And you're going to remain on that quantum state with an extra phase that is going to have important information. That's basically all that's going to happen. There's a lot of subtleties in what I've said, and we have to unmask those subtleties. But you're going to remain in that state up to a phase. That phase is going to be called something, Berry's phase. And there's a dynamical phase as well that is simple and familiar, but Berry's phase is a little less familiar.

So you're going to get the same state up to a phase. You're not going to jump states. After we'll do that, we'll do Landau-Zener transitions, which are an example where you can jump, and you will calculate and determine how big is the suppression to jump, and so it will reinforce [INAUDIBLE].

So let me begin with this thing. Suppose you have an H of t. And now, you come across this states that satisfy the following thing. H of t, psi of t, is equal to E of t psi of t. If this equation doesn't look to you totally strange, you're not looking hard enough.

It is a very strange equation. It looks familiar. It looks like everything we've always been writing, but it's not. Look what this is saying. Suppose you look at the Hamiltonian at time 0. Then the state at time 0 would be an eigenstate of the Hamiltonian at time 0 with some energy at time 0 and some state here. So this is what's called an instantaneous eigenstate.

It's an eigenstate at every time. It's almost as if you find the eigenstate at time equals 0. You find the eigenstate at time equals epsilon. You find the eigenstate at time equal 2 epsilon. Each time, and you piece together a time dependent energy eigenstate with a time dependent energy. We never did that. Our energy eigenstates were all time independent. So what kind of crazy thing is this?

Well, it has some intuition. You know how to do it. You know the Hamiltonian at every time, and at any time, you can find eigenstates. Now, you've solved at time equals 0, and you solve it at time equals epsilon, and at time equals epsilon, you're going to have different eigenstate. But at time equals 0, you're going to have lots of eigenstate. At time equals epsilon, you're going to have lots of eigenstate, but presumably, things are not changing too fast. You will know which one goes with which. Like at time equals 0, I get all these eigenstate, and at time

equal epsilon, I'll get this eigenstates, and presumably, you think, well, maybe I can join them. I'm not going to go this to that, because it's a big jump, and you can track them. So this you could find many of those. These are called are instantaneous eigenstates.

They are a little strange, because suppose you find those eigenstates, this is so far so good, but maybe this goes like that, and this crosses that one. Oh-- then how do you know which one, should you go here, or should you go here, which one is your eigenstate? So let's just hope that doesn't happen. It's going to be very difficult if it happens.

Moreover, there's going to be-- these states are not all that unique. I could multiply this Hamiltonian, this state by phase e to the i chi of t here, and a time dependent phase, the Hamiltonian wouldn't care. It would cancel. So these states are just not very unique.

Now, the more important thing I want to say about them, they are the beginning of our explicit analysis, is that do these psi's of t's solve the Schrodinger equation? Are these the solutions of the Schrodinger equation? We've found the instantaneous eigen-- so are these solutions of the Schrodinger equation? Is that what it means to solve the Schrodinger equation? I hear no. That's true. Not at all.

These are auxiliary states. They don't quite solve the Schrodinger equation. And we'll try to use them to solve the Schrodinger equation. That's what we're going to try to do. So let's try to appreciate that. This are psi's of t. Now, my notation is going to be a little delicate. Here is your Schrodinger equation. The only difference is that thing here.

Now, we're suppressing all spatial dependent. The Hamiltonian might depend on x and p, and the wave function may depend on x and p, [INAUDIBLE] x and other things, and spin, other things will just suppress them. So this is the equation we're trying to solve. This is the real equation that we're trying to solve. And if you just plug the top thing and try to see if that solves that equation, you will find it very quickly doesn't solve this equation at all.

The left hand side, if you plugged in there will appear a psi dot. If you thought psi of t solves this equation, you will have a psi dot, and here you will have an energy, and what is supposed to be a psi dot, it's not obvious. It just doesn't solve it. So on the other hand, we can try to inspire ourselves to solve it this way. You will write in ansatz.

So we'll put a psi of t. We'll try to build our solution by putting maybe the kind of thing that you usually put for an energy eigenstate. When you have an energy eigenstate, you would put an

e to the minus i et over h bar to solve the Schrodinger equation. So let's do the same thing here.

Let's put on top of the psi of t an e to the minus i over h bar energy. But the energy depends on time so, actually, the clever thing to try to put here is an integral of the energy of time, dt prime, up to t, because the main thing of that phase is that its derivative should be the energy. So that should help. So maybe this is almost a solution of the Schrodinger equation. But that may not be the case, so let's put just in case here, a c of t that maybe we will need it in order to solve the Schrodinger equation.

So our idea is OK, we're given those instantaneous eigenstates, and let's use them to get a solution of the Schrodinger equation. Of course, if we found that this is a solution of the Schrodinger equation, we would have found that with some modification, the instantaneous eigenstates produce solutions. And that would be very nice. We will find, essentially, that that's true in the adiabatic approximation.

So let's do this calculation, which is important and gives us our first sight of the adiabatic result. So here is the psi of t. Now let's substitute into the Schrodinger equation. So I have the left hand side is left hand side is i h bar dt t of this psi would be-- first, I differentiate the c. So c dot e to the minus i h bar integral to t E dt prime, psi of t.

Plus-- now I differentiate this exponent, i h bar. So I'm sorry, I have i h bar here. When I differentiate this exponent, the i's cancel with the signs. The h bar cancels. I get an E evaluated at t. I'm differentiating with respect to time here. So I get here, nicely, E of t times the [? Hall ?] wave function again, psi of t.

And then, finally, I get plus i h bar c of t-- i h bar c of t times the exponent and the time derivative of psi minus i over h bar t E dt prime times psi of t dot. So that is the dot of the state. You can differentiate the state, means evaluating the state at t plus epsilon minus [INAUDIBLE] t divide by epsilon. So we'll write it as this psi dot in there. So what do we get here.

Let's see the right hand side, right hand side is H on the state and H on the state comes here and ignores this factor, ignores these factors. Our time dependent factors come here and produces a factor of e of t. So H on psi of t is just E of t times psi of t.

So what happens, left hand side equal to right hand side. This term cancels with this. This is

nice. That's what the energy and the instantaneous states should have done. But we're left with two more terms that then cancel. These two terms. c dot is related to psi dot. So indeed, there's no obvious way of generating a solution, because there is a term in the Schrodinger equation with psi dot that must be canceled or properly taken care of by c dot here.

So what is the equation that we have? We have this first term, but the second term should be 0. So it's a simple equation. You can cancel everything basically. The phase can be canceled the i H can be canceled. So we get c dot of t times psi of t is equal to minus c of t times psi dot.

That's what we have to solve. OK, we have to solve that. Let's see how it goes. Let's see. OK, let's try to solve this equation by sandwiching psi of t, one of those instantaneous eigenstates from the left.

So what do we get here? Well, this is just a function. It just doesn't care, and psi is supposed to be normalized state. Maybe I should have said, these are instantaneous eigenstates, and psi of t, psi of t is equal to 1. They're normalized at every instant of time. That should not be difficult to implement.

So we get here c dot of t equals minus c of t psi of t, psi dot of t. That's the kind of differential equation. It just doesn't look that bad at all. In fact, it's one of those differential equations you can solve. And the answer is c of t is equal to e to the minus 0 to t psi of t prime, psi dot of t prime, dt prime.

And that's the answer. In fact, take the derivative, and you see this is an equation of the form f dot is equal to a function of time times f. This is solved by integration. That's what it is. Yes.

AUDIENCE: [INAUDIBLE] is the base?

PROFESSOR: We will see that. That's my next point. We have here a c of t, and we have an integral. Now, it looks decaying, but it's actually a phase. Let's see that. So I want to understand what is psi, psi dot. I claim that this quantity, in fact, is purely imaginary.

Let's see why. What is this thing? This is an overlap intuitively over x of psi of x and t star d dt of psi of x and t. And this, this is a dv x integral. It's a vector integral in general. I don't have to put arrows, I think. I have d dt of psi star psi minus the psi star dt psi. This is just a little bit like pre-integration by parts, is just saying a db is d of ab minus b da.

That's an identity. Now the first term, it's a d dt of an integral, so over x. So the d dt goes out.

And you have the integral over x of psi star psi. That's the first star. And the second term is minus the integral of dx of d psi star psi. But I will write it as psi d dt of psi star here with a star there. Lots of stars. Sorry.

Can you see that? The first term they took out the derivative. The second term, the sign is out, and you have the psi star times psi, but that's a complex conjugate of that other integral. The integral over space is 1. So this is 0. So this is equal to minus psi, psi dot star.

That thing in parentheses is psi dot. So what did you show that this complex number is minus its complex conjugate? So that thing is indeed a purely imaginary number. OK, so we're almost there. So let's write this nicely. What did we find?

Psi of t is equal-- the psi of t, it's c of t times this phase. So I'll put first this phase here. I'll call it c of 0 here. I ignored it before, but I could have put it here. I don't have to put it-- there's no need for it. e to the minus i over h bar, the integral from 0 to t of e of t prime dt prime. That's it.

And then, I have this factor, this c of t that I have to include. So let's put that phase here, too. It's e to the-- this minus is coming in to replace it by an i with another i in here. Why would you do that? Well, it's good notation actually, psi, psi dot of t prime, dt prime.

And all that multiplying, the instantaneous eigenstate. You see, this thing, this i in front is telling you that if you're using good notation, that this quantity is a pure phase. And indeed, it's a pure phase, because this thing is already known to be imaginary. So with an i, this is real, and with this i, this is a pure phase. So it's just notation. So here it is.

We did it. But I must say, we made a very serious mistake, and I want to know if you can identify where was our mistake. Let's give a little turn to somebody else to see where is the mistake, and then you have your go. Anybody wants to say what is the mistake. the mistake is so serious that I don't really have the right to-- look, if I didn't make a mistake, I've done something unbelievable. I found the solution of the Schrodinger equation using the instantaneous eigenstate.

I took the instantaneous eigenstate, and now I've built the solution of the Schrodinger equation. That is an unbelievable statement. It would show that you will remain in the instantaneous eigenstate forever, and I never used slow variation. So this would be an exact state. This better be wrong. This cannot be right. It cannot be that you always remain the same eigenstate. Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: That's right. There's going to be a problem with that equation. We did a little mistake here. Well, we didn't do a mistake, but we didn't do our full job. Remember in perturbation theory when you had to find the first order of correction to the state, you put from the left a state in the original subspace. You put the state outside sub space. Here we dotted with psi of t. But we have to dot with every state in the Hilbert space to make sure we have a solution.

If you have a vector equation, you cannot just dot with something and say, OK, I solved it. You might have solved the x component of the equation. So we really did not solve this equation. So we made a serious mistake in doing this. But the good thing is that this is not a bad mistake in the sense of learning. The only thing I have to say here is that this is approximately true when the changes are the adiabatic, if the change is adiabatic.

And that is what we're going to justify next time with another detailed analysis of this. So we did a good effort to find an exact solution of the Schrodinger equation, and we came close. And this is a pretty good approximation. This is the statement of the adiabatic theorem. You pretty much follow the instantaneous eigenstate up to a dynamical phase and up to a Berry phase. But this is not an exact solution of the Schrodinger equation, and in some cases, there will be transitions between those instantaneous eigenstates.