MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 18, 2012
Prof. Alan Guth

QUIZ 1

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THE FORMULA SHEETS ARE AT THE END OF THE EXAM.

____
Your Name

Problem	Maximum	Score
1	30	
2	20	
3	20	
4	30	
TOTAL	100	

* The errors that were corrected on the blackboard during the quiz are incorporated here into the text.

PROBLEM 1: SOME SHORT EXERCISES *(30 points)*

(a) *(10 points)* Use index notation to derive a formula for $\vec{\nabla} \times (s\vec{A})$, where s is a scalar field $s(\vec{r})$ and $\vec{A}$ is a vector field $\vec{A}(\vec{r})$.

(b) *(10 points)* Which of the following vector fields could describe an electric field in electrostatics? Say yes or no for each, and give a very brief reason.

(i) $\vec{E}(\vec{r}) = x\,\hat{e}_x - y\,\hat{e}_y$.

(ii) $\vec{E}(\vec{r}) = y\,\hat{e}_x + x\,\hat{e}_y$.

(iii) $\vec{E}(\vec{r}) = y\,\hat{e}_x - x\,\hat{e}_y$.

(c) *(10 points)* Suppose that the entire x-z and y-z planes are conducting. Calculate the force $\vec{F}$ on a particle of charge q located at $x = x_0$, $y = y_0$, $z = 0$.

PROBLEM 2: ELECTRIC FIELDS IN A CYLINDRICAL GEOMETRY *(20 points)*

A very long cylindrical object consists of a solid inner cylinder of radius a, which has a uniform charge density ρ, and a concentric thin cylinder, of radius $b > a$, which has an equal but opposite total charge, uniformly distributed on the surface.

(a) *(7 points)* Calculate the electric field everywhere.

(b) *(6 points)* Calculate the electric potential everywhere, taking $V = 0$ on the outer cylinder.

(c) *(7 points)* Calculate the electrostatic energy per unit length of the object.

PROBLEM 3: MULTIPOLE EXPANSION FOR A CHARGED WIRE *(20 points)*

A short piece of wire is placed along the z-axis, centered at the origin. The wire carries a total charge Q, and the linear charge density λ is an even function of z: $\lambda(z) = \lambda(-z)$. The rms length of the charge distribution in the wire is l_0; i.e.,

$$l_0^2 = \frac{1}{Q}\int_{wire} z^2 \lambda(z)\,\mathrm{d}z \ .$$

(a) *(10 points)* Find the dipole and quadrupole moments for this charge distribution. Note that the dipole and quadrupole moments are defined on the formula sheets as

$$p_i = \int d^3x\, \rho(\vec{r})\, x_i \ ,$$

$$Q_{ij} = \int d^3x\, \rho(\vec{r})(3x_i x_j - \delta_{ij}|\vec{r}|^2) \ .$$

(b) *(10 points)* Give an expression for the potential $V(r,\theta)$ for large r, including all terms through the quadrupole contribution.

PROBLEM 4: A SPHERICAL SHELL OF CHARGE *(30 points)*

(a) *(10 points)* A spherical shell of radius R, with an unspecified surface charge density, is centered at the origin of our coordinate system. The electric potential on the shell is known to be

$$V(\theta, \phi) = V_0 \sin\theta \cos\phi \ ,$$

where V_0 is a constant, and we use the usual polar coordinates, related to the Cartesian coordinates by

$$x = r \sin\theta \cos\phi \ ,$$

$$y = r \sin\theta \sin\phi \ ,$$

$$z = r \cos\theta \ .$$

Find $V(r, \theta, \phi)$ everywhere, both inside and outside the sphere. Assume that the zero of V is fixed by requiring V to approach zero at spatial infinity. (*Hint:* this problem can be solved using traceless symmetric tensors, or if you prefer you can use standard spherical harmonics. A table of the low-ℓ Legendre polynomials and spherical harmonics is included with the formula sheets.)

(b) *(10 points)* Suppose instead that the potential on the shell is given by

$$V(\theta, \phi) = V_0 \sin^2\theta \sin^2\phi \ .$$

Again, find $V(r, \theta, \phi)$ everywhere, both inside and outside the sphere.

(c) *(10 points)* Suppose instead of specifying the potential, suppose the surface charge density is known to be

$$\sigma(\theta, \phi) = \sigma_0 \sin^2\theta \sin^2\phi \ .$$

Once again, find $V(r, \theta, \phi)$ everywhere.

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FORMULA SHEET FOR QUIZ 1

Exam Date: October 18, 2012

*** Some sections below are marked with asterisks, as this section is. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

Index Notation:

$$\vec{A} \cdot \vec{B} = A_i B_i \,, \qquad \vec{A} \times \vec{B}_i = \epsilon_{ijk} A_j B_k \,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j \,, \qquad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ: $$R_z(\phi)_{ij} = \begin{array}{c c} & \begin{array}{ccc} j{=}1 & j{=}2 & j{=}3 \end{array} \\ \begin{array}{c} i{=}1 \\ i{=}2 \\ i{=}3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{n}$ by ϕ:***

$$R(\hat{n}, \phi)_{ij} = \delta_{ij} \cos\phi + \hat{n}_i \hat{n}_j (1 - \cos\phi) - \epsilon_{ijk} \hat{n}_k \sin\phi \,.$$

Vector Calculus:

Gradient: $(\vec{\nabla}\varphi)_i = \partial_i \varphi \,, \qquad \partial_i \equiv \dfrac{\partial}{\partial x_i}$

Divergence: $\vec{\nabla} \cdot \vec{A} \equiv \partial_i A_i$

Curl: $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k$

Laplacian: $\nabla^2 \varphi = \vec{\nabla} \cdot (\vec{\nabla}\varphi) = \dfrac{\partial^2 \varphi}{\partial x_i \partial x_i}$

Fundamental Theorems of Vector Calculus:

Gradient: $$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\varphi \cdot \mathrm{d}\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$$

Divergence: $$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{A}\, \mathrm{d}^3 x = \oint_S \vec{A} \cdot \mathrm{d}\vec{a}$$

where S is the boundary of $\mathcal{V}$

Curl: $$\int_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{a} = \oint_P \vec{A} \cdot \mathrm{d}\vec{\ell}$$

where P is the boundary of S

Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{r})\delta^3(\vec{r}-\vec{r}\,')\,\mathrm{d}^3x = \varphi(\vec{r}\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad g(x_i) = 0$$

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\delta^3(\vec{r}-\vec{r}\,')$$

Electrostatics:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{r}-\vec{r}\,')\,q_i}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\rho(\vec{r}\,')\,\mathrm{d}^3x'$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot \mathrm{d}\vec{\ell}\,' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\mathrm{d}^3x'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{E} = 0\,, \qquad \vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0 \ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\vec{r})V(\vec{r}) = \frac{1}{2}\epsilon_0\int \left|\vec{E}\right|^2\mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j \ , \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j \ , \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \left\{ \begin{matrix} \cos \alpha x \\ \sin \alpha x \end{matrix} \right\} \left\{ \begin{matrix} \cos \beta y \\ \sin \beta y \end{matrix} \right\} \left\{ \begin{matrix} \cosh \gamma z \\ \sinh \gamma z \end{matrix} \right\} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

Separation of Variables for Laplace's Equation in Spherical Coordinates:

Traceless Symmetric Tensor expansion:

$$\nabla^2 \varphi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\nabla_\theta^2 \varphi = 0 \ ,$$

where the angular part is given by

$$\nabla_\theta^2 \varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\nabla_\theta^2 \, C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} = -\ell(\ell+1) C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$\hat{n} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3$.

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ , \quad \text{where } \vec{r} = r\hat{n}$$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) c_\ell \{ \, \hat{z}_{i_1} \dots \hat{z}_{i_\ell} \, \} \, \hat{n}_{i_1} \dots \hat{n}_{i_\ell}$$

where $\{ \dots \}$ denotes the traceless symmetric part of $\dots$.

Special cases:

$$\{ \, 1 \, \} = 1$$

$$\{ \, \hat{z}_i \, \} = \hat{z}_i$$

$$\{ \, \hat{z}_i \hat{z}_j \, \} = \hat{z}_i \hat{z}_j - \tfrac{1}{3} \delta_{ij}$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \, \} = \hat{z}_i \hat{z}_j \hat{z}_k - \tfrac{1}{5} \big(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij} \big)$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \, \} = \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \tfrac{1}{7} \big(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij} \big) + \tfrac{1}{35} \big(\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk} \big)$$

Legendre Polynomial / Spherical Harmonic expansion:

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} \, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

Orthonormality: $\displaystyle \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, Y^*_{\ell' m'}(\theta,\phi) \, Y_{\ell m}(\theta,\phi) = \delta_{\ell' \ell} \delta_{m' m}$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Multipole Expansion:

First several terms:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\, \frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{1}{2} \frac{\hat{r}_i \hat{r}_j}{r^3} \, Q_{ij} + \cdots \right] \text{ , where}$$

$$Q = \int \mathrm{d}^3x \, \rho(\vec{r}) \, , \quad p_i = \int d^3x \, \rho(\vec{r}) \, x_i \quad Q_{ij} = \int d^3x \, \rho(\vec{r}) (3 x_i x_j - \delta_{ij} |\vec{r}|^2) \, ,$$

$$\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \big[3 (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p} \big] - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{r})$$

Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ ,$$

where

$$C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}\,') \left\{ \vec{r}\,'_{i_1} \ldots \vec{r}\,'_{i_\ell} \right\} \mathrm{d}^3x'$$

Griffiths version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int {r'}^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta')\, \mathrm{d}^3x$$

where θ' = angle between $\vec{r}$ and $\vec{r}\,'$.

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta') \ , \qquad \frac{1}{\sqrt{1-2\lambda x+\lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^\ell (x^2-1)^\ell \ , \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x\, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

Spherical Harmonic version:***

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta,\phi)$$

where $q_{\ell m} = \int Y^*_{\ell m} {r'}^{\ell} \rho(\vec{r}\,')\, \mathrm{d}^3x'$

Connection Between Traceless Symmetric Tensors and Legendre Polynomials or Spherical Harmonics:

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell (\ell!)^2} \left\{ \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \right\} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$$

For $m \geq 0$,

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m} \left\{ \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \right\}$,

with $d_{\ell m} = \dfrac{(-1)^m (2\ell)!}{2^\ell \ell!} \sqrt{\dfrac{2^m\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}$, and $\hat{u}^+ = \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$

Form $m < 0$, $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$

More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)e^{im\phi}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

Legendre Polynomials:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}$$

SPHERICAL HARMONICS $Y_{lm}(\theta, \varphi)$

$l = 0$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$l = 1$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$l = 2$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\varphi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

$l = 3$

$$Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}} \sin^3\theta e^{3i\varphi}$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \sin^2\theta \cos\theta e^{2i\varphi}$$

$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}} \sin\theta (5\cos^2\theta - 1) e^{i\varphi}$$

$$Y_{30} = \sqrt{\frac{7}{4\pi}} \left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right)$$