MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 21, 2012
Prof. Alan Guth

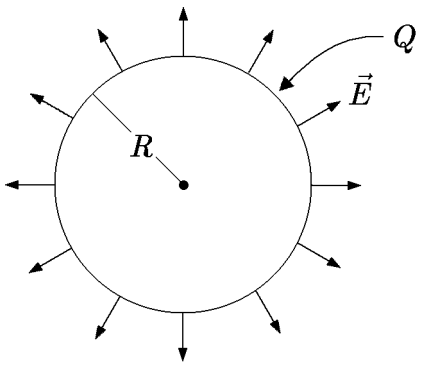
LECTURE NOTES 4
CONDUCTORS: SURFACE FORCES AND CAPACITANCE

These notes are an addendum to Lecture 7, Wednesday September 19, 2012. The notes will not repeat what I said in class, but rather will explain two topics that I did not have time to discuss in class: surface forces in conductors, and capacitance. The material in these notes will complete our class discussion of Chapter 2 of Griffiths, and it should be useful to you in completing Problem Set 2.

SURFACE FORCES COMPUTED USING VIRTUAL WORK:

The method of virtual work is simply the use of conservation of energy to determine a force — in this case, the force on the surface of a conductor. The work is *virtual* in the sense that the entire system is static, so no work is actually done. Instead one asks how much the energy of a system *would* change *if* the surface were moved an infinitesimal amount, and this change in energy is then used to compute the force.

To illustrate the method, consider a solid ball of conductor with radius R, and charge Q. The charge will flow to the surface, and by spherical symmetry the charge must distribute itself uniformly on the sphere, with surface charge density



$$\sigma = \frac{Q}{4\pi R^2} \ . \tag{4.1}$$

Our goal is to calculate the force on the surface. Since the forces depend only on the charges, the answer we get will hold for any spherical shell of charge, whether a conductor is involved or not.

To calculate the total electrostatic energy of this configuration, note that Gauss's law implies

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for } r > R \\ \vec{0} & \text{for } r < R \ . \end{cases} \tag{4.2}$$

Taking $V = 0$ at ∞, for $r > R$ we have

$$\begin{aligned} V(r) &= -\int_\infty^r \vec{E}\cdot \mathrm{d}\vec{\ell} \\ &= +\int_r^\infty \vec{E}\cdot \mathrm{d}\vec{\ell} = \frac{Q}{4\pi\epsilon_0}\int_r^\infty \frac{\mathrm{d}r}{r^2} = \frac{Q}{4\pi\epsilon_0 r} \ . \end{aligned} \tag{4.3}$$

The potential on the surface is therefore $Q/(4\pi\epsilon_0 R$, and the potential inside the sphere is constant, with this value:

$$V(r) = V(R) - \int_R^r \vec{E}\cdot \mathrm{d}\vec{\ell} = V(R) = \frac{Q}{4\pi\epsilon_0 R} \ , \tag{4.4}$$

since $\vec{E} = 0$ inside the sphere. The total electrostatic energy can be found from

$$W = \frac{1}{2}\int \rho V \, d^3x = \frac{1}{2}V(R)\int \rho d^3x = \frac{1}{2}V(R)Q = \frac{Q^2}{8\pi\epsilon_0 R} \ . \tag{4.5}$$

Now we can apply the method of virtual work. Consider a element of surface with area $\mathrm{d}a$. By spherical symmetry the force on it will be proportional to $\hat{r}$, and it must also be proportional to $\mathrm{d}a$, since $\mathrm{d}q = \sigma\,\mathrm{d}a$. So we can write the force on the surface element as

$$\mathrm{d}\vec{F} = P\,\mathrm{d}a\,\hat{r} \ , \tag{4.6}$$

where we have called the constant of proportionality P, because it is a force per area, also known as a pressure. Now imagine that the sphere is allowed to expand to radius $R + \mathrm{d}R$. The surface element will undergo a displacement $\hat{r}\,\mathrm{d}R$, so the mechanical work the electric field does on it is

$$\mathrm{d}W_{\mathrm{mech}} = \mathrm{d}\vec{F}\cdot(\hat{r}\,\mathrm{d}R) = P\,\mathrm{d}a\,\mathrm{d}R \ . \tag{4.7}$$

Integrating over the surface, the total mechanical work done is

$$\mathrm{d}W_{\mathrm{mech}} = P\,\mathrm{d}R\int \mathrm{d}a = 4\pi R^2 P\,\mathrm{d}R \ . \tag{4.8}$$

Even though we have integrated, I am still calling the work $\mathrm{d}W_{\mathrm{mech}}$, since it is an infinitesimal quantity proportional to $\mathrm{d}R$. From Eq. (4.5), the change in the total potential energy is given by

$$\mathrm{d}W = \frac{\mathrm{d}W}{\mathrm{d}R}\,\mathrm{d}R = -\frac{Q^2}{8\pi\epsilon_0 R^2}\,\mathrm{d}R \ . \tag{4.9}$$

By conservation of energy, this must equal the negative of the amount of mechanical work done. So

$$-\frac{Q^2}{8\pi\epsilon_0 R^2}\,\mathrm{d}R = -4\pi R^2 P\,\mathrm{d}R \ , \tag{4.10}$$

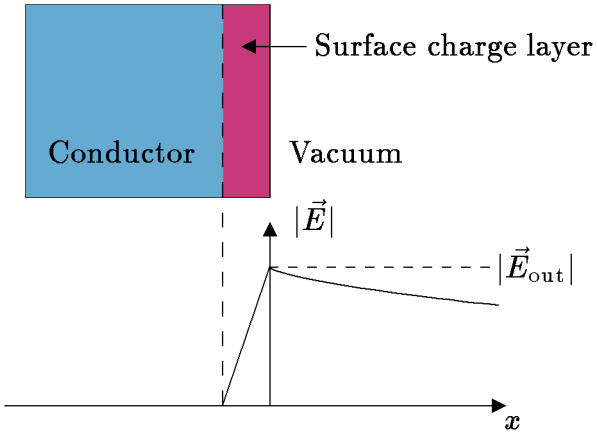
and then

$$\boxed{P = \frac{Q^2}{2(4\pi)^2\epsilon_0 R^4} = \frac{1}{2}\frac{\sigma^2}{\epsilon_0} = \frac{1}{2}\sigma|\vec{E}_{\mathrm{out}}| \ ,} \tag{4.11}$$

where $\vec{E}_{\rm out} = (\sigma/\epsilon_0)\hat{r}$ is the electric field just outside the surface.

Eq. (4.11) implies that the surface charge density σ experiences a force equivalent to an electric field $\frac{1}{2}\vec{E}_{\rm out}$, which is the average of the field outside and the vanishing electric field inside.

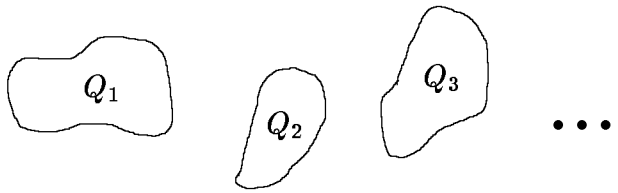
The factor of $\frac{1}{2}$ in the force law is easy to understand from a microscopic model of the charge distribution. While the approximation of an idealized conductor implies a surface charge layer of zero thickness, the idealization breaks down on the scale of atomic dimensions, so we expect that the real charge layer will have some nonzero thickness. If we model the charge density as being uniform over this thickness, the situation is described in the following diagram:



Gauss's law implies that the electric field at any point is proportional to the enclosed charge, so the electric field varies linearly with distance, as shown. (We are treating the surface as if it is a plane, since the thickness of the surface charge layer is far smaller than R.) Thus, the average piece of charge in the surface experiences an electric field that is midway between the value outside and the vanishing value inside, as we found by the method of virtual work. The description here is based on the assumption that the charge density is uniform within the surface layer, but in fact the answer does not depend on this simplification. No matter how the charge density is distributed in space, we can always divide the thin layer into N thinner layers, each containing a fraction $1/N$ of σ. We can think about the limit $N \to \infty$. Gauss's law implies that the n'th layer will experience the electric field of the first $n-1$ layers, no matter how the layers are distributed in space. Thus, the total force is independent of how the charge density is distributed in space, as long as we can assume that the thickness is small compared to R, so that the planar geometry is a valid approximation.

CAPACITANCE:

To describe capacitance in the most general possible context, we consider a system of N isolated conductors, with arbitrary shape and position:



Each conductor is an equipotential, so we can let V_i be the potential of the i'th conductor, defining the potential at $|\vec{r}| = \infty$ as zero.

If we specify the charge Q_i on the i'th conductor, for each i, and if we know the geometry of the conductors, we can in principle calculate the electric field everywhere. The equations would be

$$\vec{\nabla} \times \vec{E} = 0 \tag{4.12}$$

outside the conductors, and on the boundaries of each conductor

$$\vec{E}_\perp = \frac{\sigma}{\epsilon_0}\,\hat{n}\ , \qquad \vec{E}_\parallel = 0\ , \tag{4.13}$$

where $\vec{E}_\perp$ is the normal contribution of the electric field, $\vec{E}_\parallel$ is the tangential contribution, and $\hat{n}$ is a unit outward normal vector to the surface. The total charges on each conductor i would be specified by

$$\int_{S_i} \sigma\, \mathrm{d}a = Q_i\ . \tag{4.14}$$

The charges on each conductor will distribute themselves so that the field inside the conductor is zero, and we will prove later that this condition determines the distribution of charge uniquely. Thus, the potential on each conductor can be calculated, once all the Q_i are specified.

Suppose we put a charge Q_1 on conductor 1. The net charges on all the other conductors will remain zero, but nonetheless a surface charge density will be *induced* on the surfaces of each conductor, so that $\vec{E}$ inside each conductor will remain $\vec{0}$. If we doubled the charge on conductor 1, the linearity of the equations of electrostatics tells us that the electric field everywhere would double, and hence the potential on each conductor would double. The surface charge densities would also double.

The linearity of the equations also implies that if we put charges on two conductors, the electric field will be just the vector sum of the fields that would have been created by Q_1 and Q_2 separately. In general, then, the electric fields and hence also the potentials are linear in the charges Q_i. Thus we can write

$$V_i = \sum_j P_{ij} Q_j\ , \tag{4.15}$$

where the constants P_{ij} are determined by the geometry of the conductors. This matrix can be inverted, so we can write

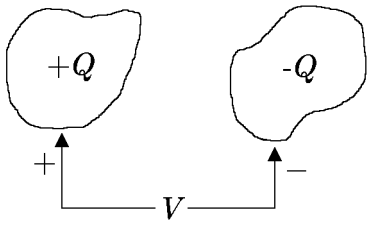
$$Q_i = \sum_j C_{ij} V_j \ , \tag{4.16}$$

where as a matrix,

$$\boldsymbol{C} = \boldsymbol{P}^{-1} \ , \text{ or equivalently } \ \sum_j C_{ij} P_{jk} = \delta_{ik} \ . \tag{4.17}$$

C_{ij} is called the *capacitance matrix*, while P_{ij} is called either the *elastance matrix*, or simply the *reciprocal capacitance matrix*. In my experience the word "elastance" is seldom used.

The most commonly discussed situation involves just two conductors, with charges that are equal in magnitude but opposite in sign. Here V is used to denote the *potential difference* between the two conductors. This pair of conductors is called a capacitor. In this case we define the *capacitance* C by



$$\boxed{Q = CV \ .} \tag{4.18}$$

Note that the charges for this case are related simply to the notation of the general case, with $Q_1 = Q$ and $Q_2 = -Q$, but the potential V is related in a more complicated way, $V = V_1 - V_2$. Thus the relation between C and the matrix C_{ij} is nontrivial — you will calculate it in Problem 5 of Problem Set 2.

The unit of capacitance is a coulomb/volt, which is defined to be a *farad*.

8.07 Electromagnetism II

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