
(16) Poisson Brackets

1 Last Time

Hamiltonian Mechanics

$$H(p, q, t) = p\dot{q} - L(q, \dot{q}, t)$$

Canonical Equations

$$\Rightarrow \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\frac{\partial H}{\partial t} = \frac{dH}{dt} \Rightarrow \frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{const}$$

and often $H = E = T + U$

2 Poisson Brackets

... how to find conserved quantities in any problem.

Let's say we have some quantity defined as a function of p , q , and t (e.g. T or U or E or L_z or whatever) and we want to know if it is a conserved quantity (constant in time). How can we tell?

Poisson Brackets

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} \\ &= \frac{\partial f}{\partial t} + [H, f]\end{aligned}$$

$$\text{where } [H, f] = \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p}$$

So, if f is conserved and not an explicit function of time, its Poisson bracket with H is zero.

$$\begin{aligned}[H, f] &= \frac{df}{dt} - \frac{\partial f}{\partial t}, \text{ so if } \frac{\partial f}{\partial t} = 0 \\ \text{then } [H, f] &= 0 \iff \frac{df}{dt} = 0\end{aligned}$$

Interestingly, for any $f(t)$, $[H, f] = 0$ simply because $\frac{\partial f}{\partial p} = \frac{\partial f}{\partial q} = 0$, or because $\frac{df}{dt} - \frac{\partial f}{\partial t} = 0$. Let's try this out...

Example: Gravity, what is $\frac{d}{dt}T$?

$$H = T + U = \frac{p^2}{2m} + mgz$$

We will need a bunch of partial derivatives, so let's compute those first...

$$\begin{aligned}\frac{\partial H}{\partial p} &= \frac{p}{m}, \quad \frac{\partial H}{\partial z} = mg, \quad \frac{\partial T}{\partial z} = 0, \quad \frac{\partial T}{\partial p} = \frac{p}{m} \\ [H, T] &= \frac{\partial H}{\partial p} \frac{\partial T}{\partial z} - \frac{\partial H}{\partial z} \frac{\partial T}{\partial p} = -gp \\ &= \frac{d}{dt}T\end{aligned}$$

Thus, the rate of change of kinetic energy is $-gp$. We can confirm this easily with conservation of energy

$$\frac{d}{dt}E = 0 \Rightarrow \frac{dT}{dt} = -\frac{dU}{dt} = -mg\dot{z} = -gp$$

The general definition of the Poisson Bracket for any two functions in an N degrees of freedom problem is

$$[f, g] = \sum_i^N \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

and it has certain properties worth knowing

$$\begin{aligned} [f, g] &= -[g, f] \quad , \quad [f, \alpha] = 0 \quad , \quad [f, f] = 0 \\ [f + g, h] &= [f, h] + [g, h] \quad (\text{distributive}) \\ [f \cdot g, h] &= f [g, h] + g [f, h] \quad (\text{product rule}) \\ \Rightarrow [f^2, g] &= 2f [f, g] \end{aligned}$$

$$\begin{aligned} [f, q_i] &= \frac{\partial f}{\partial p_i} \quad , \quad [f, p_i] = -\frac{\partial f}{\partial q_i} \quad (\text{see definition}) \\ \Rightarrow [q_j, q_k] &= 0 \quad , \quad [p_j, p_k] = 0 \quad , \quad [p_j, q_k] = \delta_{jk} \end{aligned}$$

The last and possibly most interesting of these is complicated enough to have a name

Jacobi's Identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

which I will not prove here.

This brings us to Poisson's Theorem.

Poisson's Theorem

$$\text{if } \frac{df}{dt} = 0 \quad \text{and} \quad \frac{dg}{dt} = 0 \quad \text{then} \quad \frac{d}{dt} [f, g] = 0$$

which in words is: if f and g are not explicit functions of time, and they are conserved, their Poisson Bracket is also conserved. Here is the proof:

$$\begin{aligned} \frac{df}{dt} = \frac{\partial f}{\partial t} = 0 \quad , \quad \frac{dg}{dt} = \frac{\partial g}{\partial t} = 0 &\Rightarrow [H, f] = [H, g] = 0 \\ \text{Jacobi} &\Rightarrow [H, [f, g]] + [f, [g, H]] + [g, [H, f]] = 0 \\ &\Rightarrow [H, [f, g]] = 0 \Rightarrow \frac{d}{dt} [f, g] = 0 \end{aligned}$$

Let's try a moderately complicated example of Poisson's Theorem. Consider a system in which we know that angular momentum L_x and L_y are conserved. What can we say about L_z ? Any thing?

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned}$$

$$\text{Poisson} \Rightarrow [L_x, L_y] = \text{some conserved quantity}$$

We will need some partial derivatives to compute the PB of L_x and L_y .

$$\begin{aligned} \frac{\partial L_x}{\partial \vec{r}} &= \{0, p_z, -p_y\} \quad , \quad \frac{\partial L_x}{\partial \vec{p}} = \{0, -z, y\} \\ \frac{\partial L_y}{\partial \vec{r}} &= \{-p_z, 0, p_x\} \quad , \quad \frac{\partial L_y}{\partial \vec{p}} = \{z, 0, -x\} \end{aligned}$$

$$[L_x, L_y] = \frac{\partial L_x}{\partial \vec{p}} \cdot \frac{\partial L_y}{\partial \vec{r}} - \frac{\partial L_x}{\partial \vec{r}} \cdot \frac{\partial L_y}{\partial \vec{p}} = yp_x - xp_y = -L_z$$

Thus, if L_x and L_y are conserved, so is L_z ! This is for *any* potential. (L_x and L_y must be conserved for any *initial conditions*, not *constrained*.) We already know \vec{L} is conserved for a central potential, but let me show you how to prove it with Poisson Brackets...

$$\begin{aligned} \text{if } [H, L_z] = 0 &\Rightarrow L_z = \text{const} \\ [T + U, L_z] &= [T, L_z] + [U, L_z] \end{aligned}$$

Let's do these one at a time...

$$\begin{aligned} [T, L_z] &= [T, xp_y - yp_x] = [T, xp_y] - [T, yp_x] \\ &= ([T, x]p_y + [T, p_y]x) - ([T, y]p_x + [T, p_x]y) \end{aligned}$$

It doesn't look like we are winning, but what I am doing is breaking this down into small enough parts that I can use my identities. For instance,

$$\begin{aligned} [T, p_x] &= \frac{1}{2m} [p_x^2 + p_y^2 + p_z^2, p_x] \\ [p_x^2, p_x] &= 2p_x [p_x, p_x] = 0 \\ \Rightarrow [T, p_i] &= 0 \quad \forall i \end{aligned}$$

since $T = \frac{p^2}{2m}$ contains no q_i , i.e. $\frac{\partial T}{\partial q} = 0$ and $[p_j, p_k] = 0$ (since $\frac{\partial p}{\partial q} = 0$), and

$$[T, x] = \frac{1}{2m} [p_x^2, x] = \frac{p_x}{m} [p_x, x] = \frac{p_x}{m}$$

note that $[p_y, x] = [p_z, x] = 0$. Finally,

$$[T, L_z] = \frac{p_x p_y}{m} - \frac{p_y p_x}{m} = 0$$

$\Rightarrow L_z$ conserved for free particle! (and L_x and L_y)

Well, I guess we knew that. Let's do U ...

$$\begin{aligned} [U, L_z] &= ([U, x]p_y + [U, p_y]x) - ([U, y]p_x + [U, p_x]y) \\ &= [U, p_y]x - [U, p_x]y \end{aligned}$$

where I have dropped 2 terms since $[q_j, q_k] = 0$ and $U(r)$ has no p_i in it.

$$\begin{aligned} [U, p_y] &= -\frac{\partial U}{\partial y} = -\frac{\partial U}{\partial r} \frac{\partial r}{\partial y} \\ &= -\frac{\partial U}{\partial r} \frac{y}{r} \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial y} = \frac{1}{2r} 2y = \frac{y}{r}$

Putting these together to find the PB of L_z with U ,

$$\begin{aligned} [U, L_z] &= [U, p_y]x - [U, p_x]y \\ &= -\frac{\partial U}{\partial r} \left(\frac{y}{r}x - \frac{x}{r}y \right) = 0 \end{aligned}$$

That is how Poisson Bracket manipulation works. Break it down until you hit an identity and do your best to *never* actually compute the derivatives.

For those of you who have taken 8.04, all of this should look VERY familiar. Poisson Brackets are the commutators of classical mechanics, and they work in an analogous manner. For those of you who will take 8.04 soon, remember this, because much of QM hinges on commutators!

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