Lecture (2)

Today:

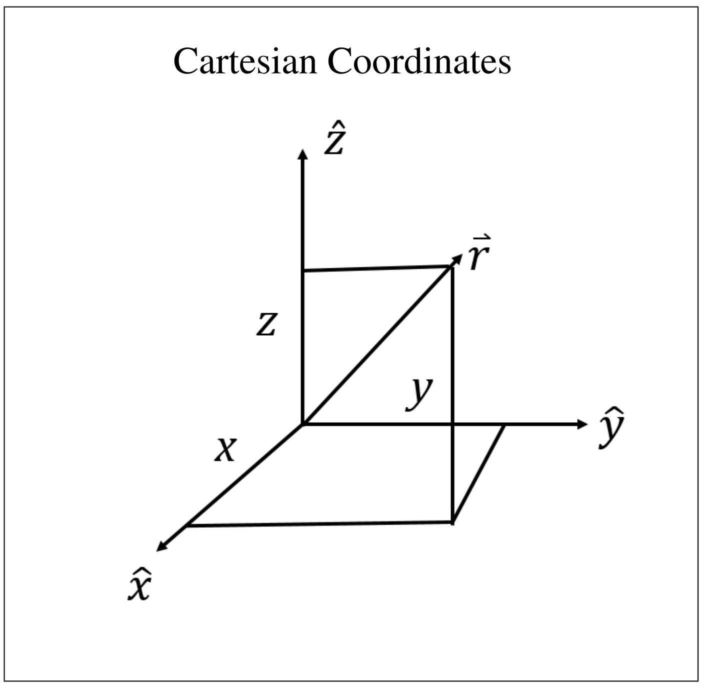
- Generalized Coordinates
- Principle of Least Action

For tomorrow

1. read LL 1-5 (only a few pages!)
2. do pset problems 4-6

1 Generalized Coordinates

The first step in almost any mechanics problem is a choice of coordinates with which to describe the motion of our system.

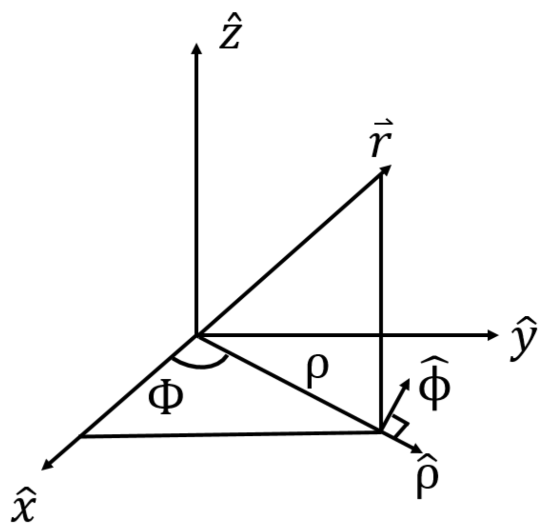


$$\vec{r} = x\,\widehat{x} + y\,\widehat{y} + z\,\widehat{z}$$

where $\widehat{x}, \widehat{y}, \widehat{z}$ are direction unit vectors.

in 3D, we need 3 parameters.

Cylindrical Coordinates

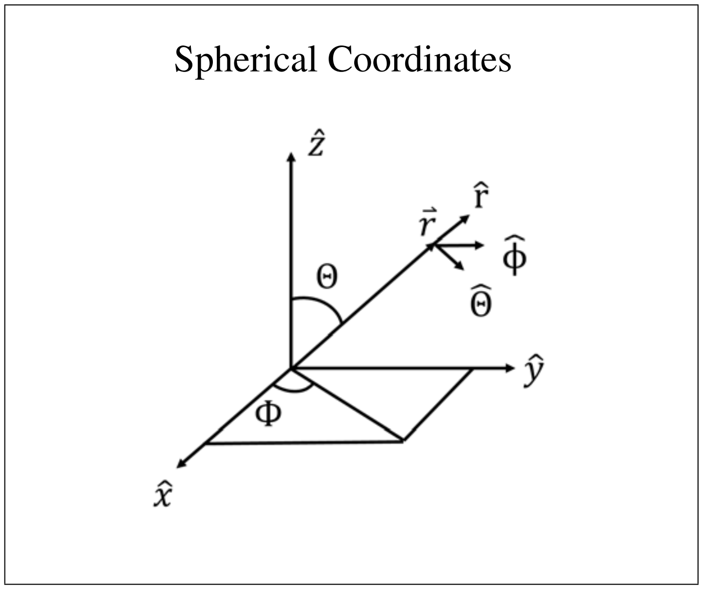


$$\rho = \sqrt{x^2 + y^2}$$
$$\cos\phi = x\ , \ \ \sin\phi = y$$
$$\vec{r} = \rho\,\widehat{\rho} + z\,\widehat{z}$$

note: $\widehat{\rho}$ and $\widehat{\phi}$ depend on ϕ

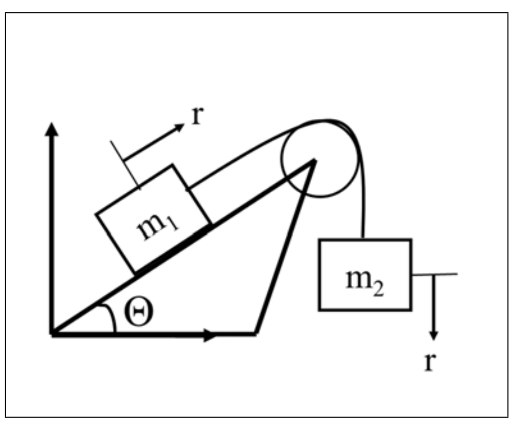
$$\Rightarrow\ \ \vec{r} = \rho\,\widehat{\rho}(\phi) + z\,\widehat{z}$$

where $\widehat{\rho}(\phi) = \rho\cos\phi\,\widehat{x} + \rho\sin\phi\,\widehat{y}$



$$
\begin{aligned}
\vec{r} &= r\,\widehat{r} \\
\widehat{r} &= \sin\theta(\cos\phi\,\widehat{x} + \sin\phi\,\widehat{y}) + \cos\theta\,\widehat{z} \\
\widehat{\theta} &= \cos\theta(\cos\phi\,\widehat{x} + \sin\phi\,\widehat{y}) - \sin\theta\,\widehat{z} \\
\widehat{\phi} &= -\sin\phi\,\widehat{x} + \cos\phi\,\widehat{y}
\end{aligned}
$$

So no matter how you do it, you need 3 coordinates for each particle in 3D. In our simple example problem, we worked with 2 masses in 2D.



2 masses in 2D
 $\Rightarrow$ 4 "Degrees of Freedom" or DOF
add 2 *constraints* (string, ramp)
 $\Rightarrow$ 2 DOF remain
$F_{x_2} = 0$, so this DOF is uninteresting
 $\Rightarrow$ 1 DOF problem (coordinate r)

So to describe the motion of this system we need only one "coordinate". I did this with a parameter called r (I tried d at first but $\frac{\partial T}{\partial d}$ looked bad!)

In the above example, we defined a coordinate r which made sense for that problem, but we did it in a creative and ill defined way... it is just what seemed to work. In order to give us a general way to talk about the coordinates with which we describe a system, we will define notation for "Generalized Coordinates".

Generalized Coordinates
for N DOFs we use $\{q_1, q_2, ..., q_N$
time derivatives are $\dot{q}_i = \frac{d}{dt} q_i$
Coordinates of a particle are implicit functions of time. We are looking for q_i(t).

I say "implicit" because we don't know the functional form until we find and solve the equations of motion. This is as opposed to an explicit time dependence, like an oscillating driving force.

Generalized coordinates needn't be orthogonal, they just need to specify the positions of all particles completely and they should be *continuously* differentiable (aka "smooth").

System State
Completely specified by q_i and $\dot{q}_i$ at any time t.
Initial state $q(t = 0)$ and $\dot{q}(t = 0)$
evolve with $F(q, \dot{q}) = m\ddot{q}$ (equation of motion).

Questions about generalized coordinates?

Note: q without subscript means $\vec{q} = \{q_i \forall i$, so 1 DOF and N DOF look the same.

2 Generalization, Take 2

Now with generalized coordinates as a well defined concept, we return to where we left off yesterday; generalizing $F = ma$. First, the generalization you did in 8.01:

Generalizing $F = ma$

explicit form $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$

let m vary $\vec{F} = \frac{d}{dt}(m(t)\dot{\vec{r}}) = \dot{\vec{p}}$

conservative force? $\vec{F} = -\nabla U = \dot{\vec{p}}$

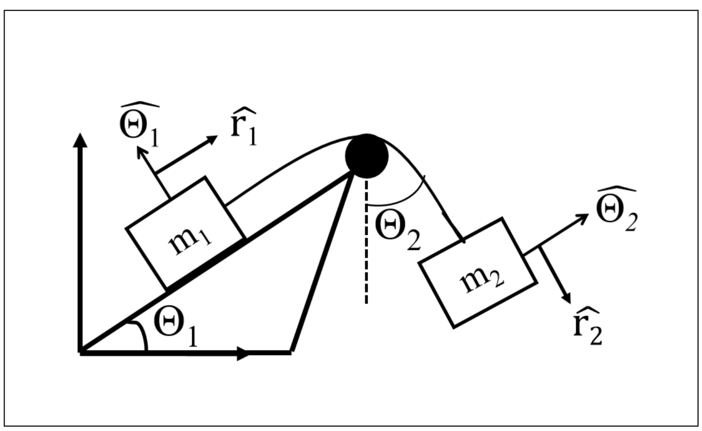
And the generalization we did yesterday, which went a little further:

$$\dot{p} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) \;,\; T = \frac{1}{2}m\dot{q}^2$$
$$\Rightarrow -\frac{\partial U}{\partial q} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)$$

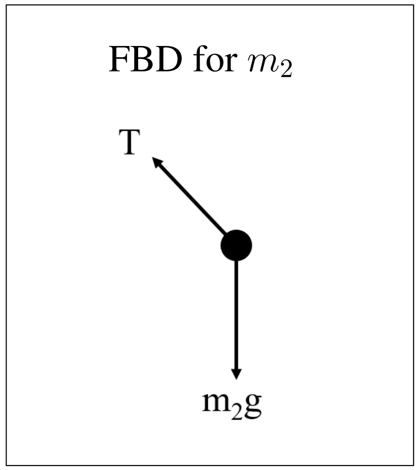
This was tricky, but it worked!

How do we know that a particular generalization is the RIGHT generalization? It's right if it lets us solve more complicated problems with ease (i.e. we use $F = \dot{p}$ for rockets with $m(t)$).

Let's try a slightly more complicated version of the ramp with two masses... we'll let the hanging mass swing.



as before, but we start with $\theta_2 \neq 0$. This means that we now have a 2 DoF system. We'll try to find the EoM first this with a simple $F = ma$ approach:



$$
\begin{aligned}
r_2\ddot{\theta}_2 &= \frac{\vec{F}_2 \cdot \widehat{\theta}_2}{m_2} \text{ or } \ddot{\theta}_2 = \frac{\tau_2}{I_2} = \frac{r_2(\vec{F}_2 \cdot \widehat{\theta}_2)}{r_2^2 m_2} \\
\ddot{\theta}_2 &= \frac{-g\sin\theta_2}{r_2} \text{ EoM for } \theta_2 \\
\ddot{r}_2 &= \frac{\vec{F}_2 \cdot \widehat{r}_2}{m_2} = \frac{\cos\theta_2 m_2 g - T_s}{m_2}
\end{aligned}
$$

Using the string tension from lecture 1,

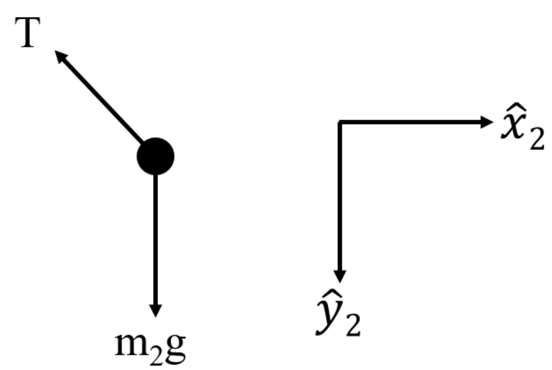
$$
T_s = \frac{m_1 m_2}{m_1 + m_2} g(\cos\theta_2 + \sin\theta)
$$

we can get

$$
\begin{aligned}
\ddot{r}_2 &= g(\cos\theta_2 - \frac{m_1}{m_1+m_2}(\cos\theta_2 + \sin\theta_1)) \\
&= \frac{g}{m_1+m_2}(m_2\cos\theta_2 - m_1\sin\theta_1) \text{ EoM for } r_2
\end{aligned}
$$

BOTH ARE WRONG! since $\widehat{r}_2(\theta_2)$ and $\widehat{\theta}_2(\theta_2)$

try Cartesian coordinates for m_2



Find EoM for x and y

$$
\begin{aligned}
m_2\ddot{x}_2 &= -T_s\sin\theta_2 \ , \ m_2\ddot{y}_2 = m_2g - T_s\cos\theta_2 \\
y_2 &= r_2\cos\theta_2 \\
\Rightarrow \ddot{y}_2 &= \frac{d}{dt}(\dot{r}_2\cos\theta_2 - r_2\dot{\theta}_2\sin\theta_2) \\
&= \ddot{r}_2\cos\theta_2 - 2\dot{r}_2\dot{\theta}_2\sin\theta_2 - r_2(\ddot{\theta}_2\sin\theta_2 + \dot{\theta}_2^{\,2}\cos\theta_2) \\
x_2 &= r_2\sin\theta_2 \\
\Rightarrow \ddot{x}_2 &= \ddot{r}_2\sin\theta_2 + 2\dot{r}_2\dot{\theta}_2\cos\theta_2 + r_2(\ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^{\,2}\sin\theta_2)
\end{aligned}
$$

returning to Equation of Motion for r_2 and θ_2

$$
\begin{aligned}
m_2(\ddot{r}_2\sin\theta_2 + \ldots) &= -T_s\sin\theta_2 \\
m_2(\ddot{r}_2\cos\theta_2 - \ldots) &= m_2g - T_s\cos\theta_2 \\
\text{recall } m_1\ddot{r}_1 &= T_s - m_1g\sin\theta_1, \ \ddot{r}_1 = \ddot{r}_2
\end{aligned}
$$

now we have everything in terms of r_2 and θ_2, we just need to solve! Ack!

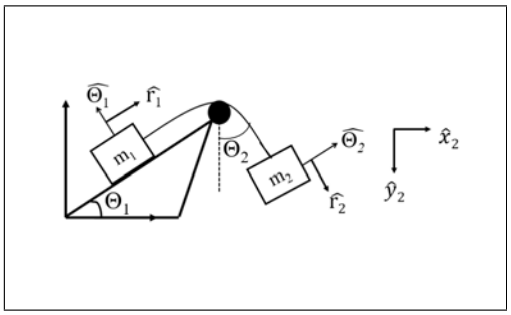
$$\Rightarrow \text{ Mathematica! } \Leftarrow$$

$$\ddot{r_2} = \frac{g}{m_1 + m_2}(m_2 \cos\theta_2 - m_1 \sin\theta_1) + \frac{m_2}{m_1 + m_2} r_2 \dot{\theta_2}^2$$

$$\ddot{\theta_2} = \frac{-1}{r_2}(g \sin\theta_2 + 2\dot{r_2}\dot{\theta_2})$$

the new term in each equation is the centripetal force!

Is this better with our generalization?



let m_2 swing... we need Potential Energy U and Kinetic Energy T to start:

$$U = g(m_1 \sin\theta_1 - m_2 \cos\theta_2)r$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_2(r\dot{\theta_2})^2$$

$$\frac{\partial U}{\partial r} = g(\quad)\ , \ \frac{\partial U}{\partial \theta_2} = g m_2 r \sin\theta_2$$

$$\frac{\partial T}{\partial \dot{r}} = (m_1 + m_2)\dot{r}\ , \ \frac{\partial T}{\partial \dot{\theta_2}} = m_2 r^2 \dot{\theta_2}$$

$$\ddot{r} = \frac{g}{m_1 + m_2}(m_2 \cos\theta_2 - m_1 \sin\theta_1)$$

This is WRONG! Centripetal force is still missing!!

$$
\begin{aligned}
\frac{d}{dt}(r^2\dot{\theta}_2) &= -gr\sin\theta_2 \\
2r\dot{r}\dot{\theta}_2 + r^2\ddot{\theta}_2 &= -gr\sin\theta_2 \\
\ddot{\theta}_2 &= \frac{-1}{r}(g\sin\theta_2 + 2\dot{r}\dot{\theta}_2)
\end{aligned}
$$

but this is now correct!

Our problem arises from the dependence of the kinetic energy on the coordinate r. In generalizing of $F = ma$, we tacitly assumed:

$$
\begin{gathered}
\text{Assumed } U(q), T(\dot{q}) \\
\text{got } \tfrac{d}{dt}(\tfrac{\partial T}{\partial \dot{q}}) = -\tfrac{\partial U}{\partial q}
\end{gathered}
$$

why not

$$
\begin{gathered}
\tfrac{d}{dt}(\tfrac{\partial L}{\partial \dot{q}}) = \tfrac{\partial L}{\partial q} \text{ for } L = T - U \\
\text{since } \tfrac{\partial U}{\partial \dot{q}} = 0 \;\text{ and }\; \tfrac{\partial T}{\partial q} = 0 \text{ ?}
\end{gathered}
$$

This seems unnecessary but let's try it...

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_2(r\dot{\theta}_2)^2 - g(m_1\sin\theta_1 - m_2\cos\theta_2)r \\
\frac{\partial L}{\partial r} &= m_2 r\dot{\theta}_2{}^2 + g(m_1\sin\theta_1 - m_2\cos\theta_2) \\
\frac{\partial L}{\partial \dot{r}} &= (m_1 + m_2)\dot{r} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\
\Rightarrow \ddot{r} &= \frac{m_2}{m_1 + m_2} r\dot{\theta}_2{}^2 + \frac{g}{m_1 + m_2}(m_1\sin\theta_1 - m_2\cos\theta_2)
\end{aligned}
$$

Done! No Mathematica needed!

Generalizations are a bit mysterious. Done randomly, most will fail to bear fruit, but the RIGHT one can open entirely new ways of approaching physics. We will take one more great leap to the Principle of Least Action, and then work our way back to Euler Lagrange and $F = ma$ from there.

Principle of Least Action (PLA):
for some $L(q, \dot{q}, t)$, the motion of a system minimizes
$S = \int_{t_1}^{t_2} L\, dt$, where $S =$ "action"
for a given $q(t_1)$ *and* $q(t_2)$

S is the action, and L is the Lagrangian. In the most general case L need not be $T - U$. But in most interesting cases, $L = T - U$.

The PLA is not like Newtonian thinking. You assume that you *know the end points*, and ask what happened in between. With $F = ma$ you assume you know the initial position *and velocity*, and then move forward in time.

For the PLA, any *trial path* is valid. The one with minimal S is the *true path*.