8.251 Final Exam

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Only personal 3-page notes allowed.
Formula sheets on the last two pages.
Test duration: 3 hours.

PROBLEM 1. (25 points) Quantum version of the rotating open string.

We have learned that in the classical theory a rigidly rotating open string has angular momentum $J = \alpha' M^2$. In this problem we explore how this relation is modified for general states in the quantum theory.

We focus on the (Hermitian) angular momentum operator J in the (x^2, x^3) plane:

$$J = M^{23} = -i \sum_{n=1}^{\infty} \frac{1}{n} \big(\alpha_{-n}^{(2)} \alpha_n^{(3)} - \alpha_{-n}^{(3)} \alpha_n^{(2)} \big) .$$

In the relation

$$\alpha' M^2 + 1 = N^{\perp} ,$$

we separate out the contributions from the the x^2 and x^3 directions by writing

$$N^{\perp} = N_{23} + N' , \quad N_{23} = \sum_{n=1}^{\infty} \big(\alpha_{-n}^{(2)} \alpha_n^{(2)} + \alpha_{-n}^{(3)} \alpha_n^{(3)} \big) ,$$

where N' denotes the number operator for the other transverse directions.

To facilitate our analysis we introduce new oscillators α_n and $\bar{\alpha}_n$ defined as

$$\alpha_n \equiv \frac{1}{\sqrt{2}} (\alpha_n^{(2)} + i \alpha_n^{(3)}) , \quad \bar{\alpha}_n \equiv \frac{1}{\sqrt{2}} (\alpha_n^{(2)} - i \alpha_n^{(3)}) .$$

Note that $(\alpha_n)^{\dagger} = \bar{\alpha}_{-n}$.

(a) Give the commutators of the α_n and $\bar{\alpha}_n$ oscillators. Rewite J in terms of the α_n and $\bar{\alpha}_n$ oscillators. Rewite N_{23} in terms of the α_n and $\bar{\alpha}_n$ oscillators.

(b) Consider general states of the theory, now written in terms of the α_n and $\bar{\alpha}_n$ oscillators:

$$|\lambda\rangle = \ldots \prod_{k=1}^{\infty} (\alpha_{-k})^{\lambda_k} (\bar{\alpha}_{-k})^{\bar{\lambda}_k} |p^+, \vec{p}_T\rangle$$

where λ_k and $\bar{\lambda}_k$ are arbitrary positive integers and the dots represent products of oscillators in directions other than two and three. Give the eigenvalues of J and N_{23} on the state $|\lambda\rangle$.

(c) Formulate and prove an inequality that relates the eigenvalues of J and those of $1 + \alpha' M^2$. [Suggestion: begin by comparing the eigenvalues of J to those of N_{23}.]

(d) Calculate both J and $1 + \alpha' M^2$ for the state $(\alpha_{-1})^N |p^+, \vec{p}_T\rangle$.

PROBLEM 2. (25 points) Cosmic string in de Sitter space.

The Nambu-Goto action on a curved spacetime with metric $g_{\mu\nu}(x)$ can be written as before

$$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$$

where all dot products use the metric $g_{\mu\nu}$ (instead of the flat Minkowski metric $\eta_{\mu\nu}$)

$$\dot{X} \cdot X' = g_{\mu\nu}(X) \dot{X}^\mu X^{\nu\prime}, \quad (\dot{X})^2 = g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu, \quad (X')^2 = g_{\mu\nu}(X) X'^\mu X'^\nu .$$

We consider strings in an expanding four-dimensional de Sitter spacetime, for which the metric $g_{\mu\nu}$ can be taken to be diagonal, with values

$$g_{00} = -1, \quad g_{11} = g_{22} = g_{33} = e^{2Ht} .$$

We are taking $c = \hbar = 1$. The Hubble constant H has natural units of one over length (or one over time) so that Ht is dimensionless.

(a) Assume $X^0 \equiv t = \tau$ and write $X^\mu = \{X^0, \vec{X}\}$. Write the Nambu-Goto action in terms of t and σ derivatives of $\vec{X}$.

(b) Consider now a circular string on the (x^1, x^2) plane, namely

$$X^1(t, \sigma) = r(t) \cos\sigma, \quad X^2(t, \sigma) = r(t) \sin\sigma, \quad \sigma \in [0, 2\pi],$$

where $r(t)$ is a radius function to be determined. Use this ansatz to simplify the string action and perform the integration over σ. Write the resulting action as

$$S = \int dt\, L(\dot{r}(t), r(t); t)$$

and give the explicit form of $L(\dot{r}(t), r(t); t)$, which is explicitly time dependent. Because of the e^{2Ht} factors in the metric, the physically measured radius of the string is actually $R(t) = e^{Ht}\, r(t)$. Write the Lagrangian in terms of R and $\dot{R}$.

(c) Consider strings with constant R and use the Lagrangian to give the potential $V(R)$ for such strings. Plot this potential and verify that it is well-defined only if $R \leq 1/H$. Find a critical point of the potential and the corresponding value of R (in terms of H). Is this static string in stable equilibrium?

(d) Use the Lagrangian for R and $\dot{R}$ to calculate the corresponding Hamiltonian, expressed as a function of R and $\dot{R}$. Simplify your answer. Is this Hamiltonian function conserved for physical motion?

PROBLEM 3. Three short questions (20 points).

(a) Express the open string state

$$L_{-2}^{\perp}L_{-2}^{\perp}|0\rangle$$

in terms of normal-ordered oscillators acting on the zero-momentum ground state $|0\rangle$.

(b) State the field content of the massless spectrum of a configuration of N coincident Dp-branes (let d denote the number of spatial dimensions). What happens to the massless spectrum when we add a coincident orientifold Op-plane?

(c) Two infinite D2-branes intersect at right angles. Recalling that the light-cone gauge requires that the X^1 coordinate have Neumann boundary conditions at both ends, we take the first D2-brane to extend in the 1- and 2-directions, and the second D2-brane to extend in the 1- and 3-directions. Calculate the value of M^2 for all tachyons in the sector that contains the open strings that stretch from one brane to the other. Where do these tachyons live?

PROBLEM 4. (30 points) Counting states in Heterotic SO(32) string theory.

In heterotic (closed) string theory the right-moving part of the theory is that of an open superstring. It has an NS sector whose states are built with oscillators α_{-n}^I and b_{-r}^I acting on the NS vacuum. It also has an R sector whose states are built with oscillators α_{-n}^I and d_{-n}^I acting on the R ground states. The index I runs over 8 values. The standard GSO projection down to NS+ and R− applies.

The left-moving part of the theory is that of a peculiar bosonic open string. The 24 transverse coordinates split into eight bosonic coordinates X^I with oscillators $\bar{\alpha}_{-n}^I$ and 16 peculiar bosonic coordinates. A surprising fact of two-dimensional physics allows us to replace these 16 coordinates by 32 two-dimensional left-moving *fermion* fields λ^A, with $A = 1, 2, \ldots, 32$. The (anticommuting) fermion fields λ^A imply that the left-moving part of the theory also has NS′ and R′ sectors, denoted with primes to differentiate them from the standard NS and R sectors of the open superstring.

The left NS′ sector is built with oscillators $\bar{\alpha}_{-n}^I$ and λ_{-r}^A acting on the vacuum $|\mathrm{NS}'\rangle_L$, declared to have $(-1)^{F_L} = +1$:

$$(-1)^{F_L}|\mathrm{NS}'\rangle_L = +|\mathrm{NS}'\rangle_L\,.$$

The *naive* mass formula in this sector is

$$\alpha' M_L^2 = \frac{1}{2}\sum_{n\neq 0}\bar{\alpha}_{-n}^I\bar{\alpha}_n^I + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} r\,\lambda_{-r}^A\lambda_r^A\,.$$

The left R′ sector is built with oscillators $\bar{\alpha}_{-n}^I$ and λ_{-n}^A acting on a set of R′ ground states. The naive mass formula in this sector is

$$\alpha' M_L^2 = \frac{1}{2}\sum_{n\neq 0}\Big(\bar{\alpha}_{-n}^I\bar{\alpha}_n^I + n\,\lambda_{-n}^A\lambda_n^A\Big)\,.$$

Momentum labels are not needed in this problem so they are omitted throughout.

(a) Consider the left NS$'$ sector. Write the precise mass-squared formula with normal-ordered oscillators and the appropriate normal-ordering constant. The GSO projection here keeps the states with $(-1)^{F_L} = +1$; this defines the left NS$'$+ sector. Write explicitly and count the states we keep for the three lowest mass levels, indicating the corresponding values of $\alpha' M_L^2$. [This is a long list.]

(b) Consider the left R$'$ sector. Write the precise mass-squared formula with normal-ordered oscillators and the appropriate normal-ordering constant. We have here 32 zero modes λ_0^A. As usual, half of the ground states have $(-1)^{F_L} = +1$ and the other half have $(-1)^{F_L} = -1$. Let $|R_\alpha\rangle_L$ denote ground states with $(-1)^{F_L} = +1$. How many ground states $|R_\alpha\rangle_L$ are there? Keep only states with $(-1)^{F_L} = +1$; this defines the left R$'$+ sector. Write explicitly and count the states we keep for the two lowest mass levels, indicating the corresponding values of $\alpha' M_L^2$. [This is a shorter list.]

At any mass level $\alpha' M^2 = 4k$ of the **heterotic** string, the spacetime bosons are obtained by "tensoring" **all** the left states (NS$'$+ and R$'$+) with $\alpha' M_L^2 = k$ with the right-moving NS+ states with $\alpha' M_R^2 = k$. Similarly, the spacetime fermions are obtained by tensoring all the left states (NS$'$+ and R$'$+) with $\alpha' M_L^2 = k$ with the right-moving R$-$ states with $\alpha' M_R^2 = k$. At any mass level where either left states or right states are missing, one cannot form heterotic string states.

(c) Are there tachyonic states in heterotic string theory? Write out the massless states of the theory (bosons and fermions) and describe the fields associated with the bosons. Calculate the total number of states in heterotic string theory (bosons plus fermions) at $\alpha' M^2 = 4$. (This is a large number!)

(d) Write a generating function $f_L(x) = \sum_r a(r) x^r$ for the full set of (GSO truncated) states in the left-moving sector (include both NS$'$+ and R$'$+ states). Use the convention where $a(r)$ counts the number of states with $\alpha' M_L^2 = r$.

Possibly Useful Formulas

Light-Cone Coordinates: $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right)$.

Relativistic Point Particle in Light-Cone Coordinates: $x^+ = \frac{p^+}{m^2}\,\tau$, $\quad p^- = \frac{1}{2p^+}\left(p^I p^I + m^2\right)$.

Slope Parameter: $\alpha' = \frac{1}{2\pi T_0}$.

Light-Cone Gauge:

$$X^+ = \beta\alpha' p^+\,\tau, \quad \text{where } \beta = \begin{cases} 2 & \text{for open strings} \\ 1 & \text{for closed strings ,} \end{cases}$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\,\dot{X}^\mu \ , \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}\,X^{\mu\prime} \ ,$$

$$(\dot{X} \pm X')^2 = 0 \quad \Longrightarrow \quad \dot{X}^- \pm X^{-\prime} = \frac{1}{2\beta\alpha' p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2 \ ,$$

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0 \ .$$

Open String Expansion: $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\,e^{-in\tau}\,\cos n\sigma$.

Closed String Expansion: $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^\mu\,e^{in\sigma} + \bar{\alpha}_n^\mu\,e^{-in\sigma}\right)$.

Momentum: $\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu$ (open strings) , $\qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu$ (closed strings).

Commutators, Creation and Annihilation Operators:

$$\left[\alpha_n^I\,,\,\alpha_m^J\right] = n\delta_{m+n,0}\,\delta^{IJ} \ , \qquad \left[x_0^I\,,\,\alpha_0^J\right] = \sqrt{2\alpha'}\,i\,\delta^{IJ} \ , \qquad \left[x_0^I\,,\,\alpha_n^J\right] = 0 \text{ if } n\neq 0 \ ,$$

$$\text{for } n\geq 1: \qquad \alpha_n^I = \sqrt{n}\,a_n^I \ , \qquad \alpha_{-n}^I = \sqrt{n}\,a_n^{I\dagger} \ , \qquad \left[a_m^I\,,\,a_n^{J\dagger}\right] = \delta^{IJ}\,\delta_{mn} \ .$$

Virasoro Algebra (Open Strings):

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\,p^+}L_n^\perp \ , \qquad \text{where } L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I \ , \quad \left[L_m^\perp\,,\,\alpha_n^J\right] = -n\alpha_{m+n}^J \ ,$$

$$\left[L_m^\perp\,,\,L_n^\perp\right] = (m-n)L_{m+n}^\perp + \frac{1}{12}m(m^2-1)(D-2)\delta_{m+n,0} \ , \quad \left[L_m^\perp\,,\,x_0^I\right] = -i\sqrt{2\alpha'}\,\alpha_m^I \ ,$$

$$L_0^\perp = \alpha'\,p^I\,p^I + N^\perp \ , \qquad N^\perp = \sum_{p=1}^{\infty}\alpha_{-p}^I\,\alpha_p^I = \sum_{n=1}^{\infty} n\,a_n^{I\dagger}\,a_n^I \ ,$$

$$M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I = \frac{1}{\alpha'}\left(N^\perp - 1\right) \ .$$

Virasoro Algebra (Closed Strings):

$$\alpha_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\,L_n^\perp \ , \quad \bar{\alpha}_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\,\bar{L}_n^\perp \ , \quad L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I \ , \quad \bar{L}_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\bar{\alpha}_{n-p}^I\,\bar{\alpha}_p^I \ ,$$

$$L_0^\perp = \frac{\alpha'}{4}\,p^I\,p^I + N^\perp \ , \quad N^\perp = \sum_{p=1}^{\infty}\alpha_{-p}^I\,\alpha_p^I = \sum_{n=1}^{\infty} n\,a_n^{I\dagger}\,a_n^I \ ,$$

$$\bar{L}_0^\perp = \frac{\alpha'}{4}\,p^I\,p^I + \bar{N}^\perp \ , \quad \bar{N}^\perp = \sum_{p=1}^{\infty}\bar{\alpha}_{-p}^I\,\bar{\alpha}_p^I = \sum_{n=1}^{\infty} n\,\bar{a}_n^{I\dagger}\,\bar{a}_n^I \ ,$$

$$M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I = \frac{2}{\alpha'}\left(N^\perp + \bar{N}^\perp - 2\right) \ , \quad \bar{N}^\perp = N^\perp \ .$$

NS-sector:

Ground state: $(-1)^F = -1$: $|\mathrm{NS}\rangle$.

Normal ordering constant for NS fermion: $a_{NS} = -\frac{1}{48}$.

Mass-squared: $\alpha' M^2 = -\frac{1}{2} + N^\perp$.

$$\begin{aligned}
\alpha' M^2 = -\tfrac{1}{2}\,,\ N^\perp = 0 &: |\mathrm{NS}\rangle\,,\\
\alpha' M^2 = \ 0\,,\ N^\perp = \tfrac{1}{2} &: b^I_{-1/2}|\mathrm{NS}\rangle\,,\\
\alpha' M^2 = \ \tfrac{1}{2}\,,\ N^\perp = 1 &: \big\{\alpha^I_{-1}\,,\ b^I_{-1/2}b^J_{-1/2}\big\}|\mathrm{NS}\rangle\,,\\
\alpha' M^2 = 1\,,\ N^\perp = \tfrac{3}{2} &: \big\{\alpha^I_{-1}b^J_{-1/2}\,,\ b^I_{-3/2}\,,\ b^I_{-1/2}b^J_{-1/2}b^K_{-1/2}\big\}|\mathrm{NS}\rangle\,.
\end{aligned}$$

NS+ sector: $(-1)^F = +1$. Integer $\alpha' M^2$.

R-sector:

8 zero modes: $\to$ 4 creation + 4 annihilation. 4 creation $\to 2^4 = 16$ ground states.

Ground states: $(-1)^F = -1$: $|R_a\rangle$, $a = 1, \dots 8$, and $(-1)^F = +1$: $|R_{\bar a}\rangle$, $\bar a = \bar 1, \dots \bar 8$.

Normal ordering constant for Ramond fermion: $a_R = +\frac{1}{24}$.

Mass-squared: $\alpha' M^2 = N^\perp$.

$$\begin{array}{lr||l}
\alpha' M^2 = 0: & |R_a\rangle & |R_{\bar a}\rangle\\
\alpha' M^2 = 1: & \alpha^I_{-1}|R_a\rangle\,,\ d^I_{-1}|R_{\bar a}\rangle & \alpha^I_{-1}|R_{\bar a}\rangle\,,\ d^I_{-1}|R_a\rangle,\\
\alpha' M^2 = 2: & \{\alpha^I_{-2}\,,\ \alpha^I_{-1}\alpha^J_{-1}\,, d^I_{-1}d^J_{-1}\}|R_a\rangle & \{\alpha^I_{-2}\,,\ \alpha^I_{-1}\alpha^J_{-1}\,, d^I_{-1}d^J_{-1}\}|R_{\bar a}\rangle\\
 & \{\alpha^I_{-1}d^J_{-1}\,,\ d^I_{-2}\}|R_{\bar a}\rangle & \{\alpha^I_{-1}d^J_{-1}\,,\ d^I_{-2}\}|R_a\rangle
\end{array}$$

Left of bars: $(-1)^F = -1$, the R$-$ sector. Right of bars: $(-1)^F = +1$, the R+ sector.

$$f_{NS+}(x) = \frac{1}{2\sqrt{x}}\Big[\prod_{n=1}^\infty\Big(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\Big)^8 - \prod_{n=1}^\infty\Big(\frac{1-x^{n-\frac{1}{2}}}{1-x^n}\Big)^8\Big] = 8\prod_{n=1}^\infty\Big(\frac{1+x^n}{1-x^n}\Big)^8 = f_{R-}(x)\,.$$

Dp-brane: Oscillators: α^i_n, $i = 2, 3, \dots, p$ and α^a_n, $a = p+1, \dots, d$.

Ground states: $|p^+, \vec p\rangle$, $\vec p = (p^2, \dots p^p)$.

Mass-squared: $\alpha' M^2 = -1 + \sum_{n=1}^\infty\big(\alpha^i_{-n}\alpha^i_n + \alpha^a_{-n}\alpha^a_n\big)$.

Dp-brane and coincident Op:

Ω_p action: $\Omega_p\,\alpha^i_n\,\Omega_p^{-1} = (-1)^n\alpha^i_n$, $\Omega_p\,\alpha^a_n\,\Omega_p^{-1} = (-1)^n\alpha^a_n$, ground states invariant.

N parallel Dp-branes: Ground states $|p^+, \vec p; [j\,k]\rangle$, $j, k = 1, 2, \dots, N$, $\vec p = (p^2, \dots p^p)$.

If present, Ω_p action on the ground states $\Omega_p|p^+, \vec p; [j\,k]\rangle = |p^+, \vec p; [k\,j]\rangle$

Normal ordering constants: $a_{NN} = a_{DD} = -\frac{1}{24}$, $a_{ND} = a_{DN} = +\frac{1}{48}$.