Assignment #5 **PHYSICS 8.284**

due 11:04 am Friday 2006 March 17th

- Reading: Hansen and Kawaler (Hayden reserve only): §§3.1-3.5 (eqn 3.67) on quantum statistics, distribution functions and equations of state and §§7.1-7.2 (eqn 7.45) on polytropes.

1. Solve the Lane-Emden equation for an n=3/2 polytrope using Runge-Kutta integration from $\xi = 0$ to the point where the density first reaches zero. Plot ϕ^n (which is proportional to density) and $-4\pi\xi^2 \frac{d\phi}{d\xi}$ (which is proportional to the mass interior to ξ) as a function of ξ.

2. In class we derived an expression for the gravitational binding energy, Ω, of a polytrope of index n. Use the virial theorem to derive and expression for the total internal energy, U, of such a polytrope. Define the average temperature, $\bar{T}$, such that $U = \frac{M}{\mu m_p} \frac{3}{2} k\bar{T}$. Compute the ratio of the central temperature T_c to this average temperature.

3. (Clayton's problem 1-22) A certain stellar atmosphere with a pressure of 1000 dynes/cm^2 consists entirely of hydrogen (molecular and atomic). By ignoring statistical weights, i.e. setting the partition functions equal to unity, find the temperature at which the H_2 molecules are 50 percent dissociated into atomic hydrogen. The binding energy of the H_2 molecule is 4.48 ev. Ignore ionization. Does your result justify this approximation?

4. Consider (for the sake of argument) an infinitely long cylindrically symmetric self gravitating star in hydrostatic equilibrium which obeys a polytropic equation of state, $P = K\rho^{1+\frac{1}{n}}$. Let $\mu(r)$ be the mass *per unit length* interior to r, i.e. $\mu(r) = \int_0^r \rho(r) 2\pi r dr$.

   a) Derive the cylindrical analog of the Lane-Emden equation for such a star. (Hint: Gauss' Law applied to an infinitely long cylindrical mass gives a force $-2G\mu(r)/r$. Use this in the equation of hydrostatic equilibrium). What is the scale length (which you might call b to distinguish it from the scale length a in the spherical case) used to make the equation dimensionless.

   b) Derive an expression for $\mu(r)$ interior to the first zero of the solution, ξ_1, in terms of the central density (variously represented by λ and ρ_c) and the scale length b.

   c) For spherical stars of polytropic index $n = 3$ we found a unique mass which was independent of central density and depended only on the constant K. For a relativistic degenerate electron gas this gave us the Chandrasaekhar mass. For what polytropic index is $\mu(r)$ independent of central density?

(5) The (log T, log ρ) plane can be divided into four regions in which radiation pressure, non-degenerate gas pressure, degenerate electron pressure and relativistic degenerate electron pressure dominate the total pressure. These four regions are separated by three straight lines.

   a) By equating the expressions for pressure in adjacent regions, derive an equation for each of these straight lines.

   b) draw them on a plot of the (log ρ, log P) plane. In computing the mean molecular weight, take $X = 0.70$, $Y = 0.28$ and $Z = 0.02$, where X is the fraction by *mass* of hydrogen, Y is mass fraction of helium, and Z is the "heavy element" abundance (everything else) for which the mean molecular weight, $\mu \approx 2$.