MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 5, 2013
Prof. Alan Guth

QUIZ 3

Reformatted to Remove Blank Pages*

A FORMULA SHEET IS AT THE END OF THE EXAM.

You may rip off and keep the formula sheet.

Please answer all questions in this stapled booklet.

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Your Name

* A few corrections announced at the quiz have been incorporated.

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

(a) *(5 points)* Ryden summarizes the results of the COBE satellite experiment for the measurements of the cosmic microwave background (CMB) in the form of three important results. The first was that, in any particular direction of the sky, the spectrum of the CMB is very close to that of an ideal blackbody. The FIRAS instrument on the COBE satellite could have detected deviations from the blackbody spectrum as small as $\Delta\epsilon/\epsilon \approx 10^{-n}$, where n is an integer. To within ± 1, what is n?

(b) *(5 points)* The second result was the measurement of a dipole distortion of the CMB spectrum; that is, the radiation is slightly blueshifted to higher temperatures in one direction, and slightly redshifted to lower temperatures in the opposite direction. To what physical effect was this dipole distortion attributed?

(c) *(5 points)* The third result concerned the measurement of temperature fluctuations after the dipole feature mentioned above was subtracted out. Defining

$$\frac{\delta T}{T}(\theta,\phi) \equiv \frac{T(\theta,\phi) - \langle T \rangle}{\langle T \rangle} \ ,$$

where $\langle T \rangle = 2.725$ K, the average value of T, they found a root mean square fluctuation,

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} \ ,$$

equal to some number. To within an order of magnitude, what was that number?

(d) *(5 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

(iii) Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

(e) *(5 points)* The flatness problem refers to the extreme fine-tuning that is needed in Ω at early times, in order for it to be as close to 1 today as we observe. Starting with the assumption that Ω today is equal to 1 within about 1%, one concludes that at one second after the big bang,

$$|\Omega - 1|_{t=1 \text{ sec}} < 10^{-m} ,$$

where m is an integer. To within $\pm$ 3, what is m?

(f) *(5 points)* The total energy density of the present universe consists mainly of baryonic matter, dark matter, and dark energy. Give the percentages of each, according to the best fit obtained from the Planck 2013 data. You will get full credit if the first (baryonic matter) is accurate to $\pm 2\%$, and the other two are accurate to within $\pm 5\%$.

(g) *(5 points)* Within the conventional hot big bang cosmology (without inflation), it is difficult to understand how the temperature of the CMB can be correlated at angular separations that are so large that the points on the surface of last scattering was separated from each other by more than a horizon distance. Approximately what angle, in degrees, corresponds to a separation on the surface last scattering of one horizon length? You will get full credit if your answer is right to within a factor of 2.

PROBLEM 2: FREEZE-OUT OF MUONS *(25 points)*

The following problem was on Problem Set 7, Problem 2 (2013).

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) *(5 points)* The formula for the energy density of black-body radiation, as given by Eq. (6.48) of the lecture notes,

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ ,$$

is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) *(8 points)* When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) *(12 points)* As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting a denote the Robertson-Walker scale factor, by what factor does the quantity aT increase when the muons disappear?

PROBLEM 3: THE EVENT HORIZON FOR OUR UNIVERSE *(25 points)*

We have learned that the expansion history of our universe can be described in terms of a small set of numbers: $\Omega_{m,0}$, the present contribution to Ω from nonrelativistic matter; $\Omega_{\rm rad,0}$, the present contribution to Ω from radiation; $\Omega_{\rm vac}$, the present contribution to Ω from vacuum energy; and H_0, the present value of the Hubble expansion rate. The best estimates of these numbers are consistent with a flat universe, so we can take $k = 0$, $\Omega_{m,0} + \Omega_{\rm rad,0} + \Omega_{\rm vac} = 1$, and we can use the flat Robertson-Walker metric,

$$\mathrm{d}s^2 = -c^2\, dt^2 + a^2(t) \left[\mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right] \ .$$

(a) *(5 points)* Suppose that we are at the origin of the coordinate system, and that at the present time t_0 we emit a spherical pulse of light. It turns out that there is a maximum coordinate radius $r = r_{\rm max}$ that this pulse will ever reach, no matter how long we wait. (The pulse will never actually reach $r_{\rm max}$, but will reach all r such that $0 < r < r_{\rm max}$.) $r_{\rm max}$ is the coordinate of what is called the *event horizon*: events that happen now at $r \geq r_{\rm max}$ will never be visible to us, assuming that we remain at the origin. Assuming for this part that the function $a(t)$ is a known function, write an expression for $r_{\rm max}$. Your answer should be expressed as an integral, which can involve $a(t)$, t_0, and any of the parameters defined in the preamble. [*Advice:* If you cannot answer this, you should still try part (c).]

(b) *(10 points)* Since $a(t)$ is not known explicitly, the answer to the previous part is difficult to use. Show, however, that by changing the variable of integration, you can rewrite the expression for $r_{\rm max}$ as a definite integral involving only the parameters specified in the preamble, without any reference to the function $a(t)$, except perhaps to its present value $a(t_0)$. You are not expected to evaluate this integral. [*Hint:* One method is to use

$$x = \frac{a(t)}{a(t_0)}$$

as the variable of integration, just as we did when we derived the first of the expressions for t_0 shown in the formula sheets.]

(c) *(10 points)* Astronomers often describe distances in terms of redshifts, so it is useful to find the redshift of the event horizon. That is, if a light ray that originated at $r = r_{\rm max}$ arrived at Earth today, what would be its redshift $z_{\rm eh}$ (eh = event horizon)? You are not asked to find an explicit expression for $z_{\rm eh}$, but instead an equation that could be solved numerically to determine $z_{\rm eh}$. For this part you can treat $r_{\rm max}$ as given, so it does not matter if you have done parts (a) and (b). You will get half credit for a correct answer that involves the function $a(t)$, and full credit for a correct answer that involves only explicit integrals depending only on the parameters specified in the preamble, and possibly $a(t_0)$.

PROBLEM 4: BEHAVIOR OF Ω IN A UNIVERSE DOMINATED BY MYSTERIOUS STUFF *(15 points)*

(a) *(5 points)* In discussing the flatness problem, we learned how to calculate the behavior of $(\Omega - 1)/\Omega$, learning that it obeys an equation of the form

$$\frac{\Omega - 1}{\Omega} = A\frac{T^2}{\rho} \; , \tag{4.1}$$

where A is to a good approximation independent of time. Show how Eq. (4.1) can be derived from the first order Friedmann equation,

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \; , \tag{4.2}$$

and in the process you should find the expression for A. (No credit will be given for simply writing the expression for A from memory.)

(b) *(10 points)* Using Eq. (4.1), we learned that $(\Omega - 1)/\Omega$ grows as t in a radiation-dominated universe, and as $t^{2/3}$ in a matter-dominated universe. Suppose, however, that we consider a universe dominated by "mysterious stuff," which has the property that

$$\rho \propto \frac{1}{a^5(t)} \; . \qquad (4.3))$$

(You may recall that on Quiz 2 we had a problem concerning a hypothetical universe dominated by this fictitious material.) How would $(\Omega - 1)/\Omega$ behave in a universe dominated by mysterious stuff? You may assume that the universe is very nearly flat.

Problem	**Maximum**	**Score**
1	35	
2	25	
3	25	
4	15	
TOTAL	100	

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QUIZ 3 FORMULA SHEET

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\text{coord}} = \frac{c}{a(t)} \ .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \ ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \ ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)} \rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)} \rho_r(t_i) \ \text{(radiation)}.$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$): $a(t) \propto t^{2/3}$ (matter-dominated) ,

$a(t) \propto t^{1/2}$ (radiation-dominated) ,

$\Omega = 1$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \ , \quad \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{a}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ .$$

Alternatively, for $k > 0$, we can define $r = \dfrac{\sin\psi}{\sqrt{k}}$, and then

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{k}$. For $k < 0$ we can define $r = \dfrac{\sinh\psi}{\sqrt{-k}}$, and then

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}\ ,$$

where $\tilde{a}(t) = a(t)/\sqrt{-k}$. Note that $\tilde{a}$ can be called a if there is no need to relate it to the $a(t)$ that appears in the first equation above.

HORIZON DISTANCE:

$$\begin{aligned}\ell_{p,\text{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}\,dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases}\end{aligned}$$

SCHWARZSCHILD METRIC:

$$\begin{aligned}ds^2 = -c^2 d\tau^2 = &-\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 \\ &+ r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,\end{aligned}$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

$$\text{or:}\qquad \frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

$$g_\gamma = g_\gamma^* = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g_\nu^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} ,$$

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 = -\frac{\Lambda c^4}{8\pi G} .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2} ,$$

where

$$x \equiv \frac{a(t)}{a(t_0)} \equiv \frac{1}{1+z} ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0} .$$

Age of universe:

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}}$$

$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}}\ .$$

Look-back time:

$$t_{\text{look-back}}(z) =$$

$$\frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rad},0}(1+z')^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z')^2}}\ .$$

PHYSICAL CONSTANTS:

$$G = 6.674\times10^{-11}\ \text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2} = 6.674\times10^{-8}\ \text{cm}^3\cdot\text{g}^{-1}\cdot\text{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381\times10^{-23}\,\text{joule/K} \\ &= 1.381\times10^{-16}\,\text{erg/K} \\ &= 8.617\times10^{-5}\,\text{eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055\times10^{-34}\ \text{joule}\cdot\text{s} \\ &= 1.055\times10^{-27}\ \text{erg}\cdot\text{s} \\ &= 6.582\times10^{-16}\ \text{eV}\cdot\text{s} \end{aligned}$$

$$\begin{aligned} c &= 2.998\times10^8\ \text{m/s} \\ &= 2.998\times10^{10}\ \text{cm/s} \end{aligned}$$

$$\hbar c = 197.3\ \text{MeV-fm},\quad 1\ \text{fm} = 10^{-15}\ \text{m}$$

$$1\ \text{yr} = 3.156\times10^7\ \text{s}$$

$$1\ \text{eV} = 1.602\times10^{-19}\ \text{joule} = 1.602\times10^{-12}\ \text{erg}$$

$$\begin{aligned} 1\ \text{GeV} = 10^9\ \text{eV} &= 1.783\times10^{-27}\ \ \text{kg (where } c\equiv 1) \\ &= 1.783\times10^{-24}\ \text{g}\ . \end{aligned}$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\begin{aligned}
\ell_P &= \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-35} \text{ m} , \\
&= 1.616 \times 10^{-33} \text{ cm} , \\
t_P &= \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44} \text{ s} , \\
m_P &= \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-8} \text{ kg} , \\
&= 2.177 \times 10^{-5} \text{ g} , \\
E_P &= \sqrt{\frac{\hbar c^5}{G}} = 1.221 \times 10^{19} \text{ GeV} .
\end{aligned}$$

CHEMICAL EQUILIBRIUM:

(This topic was not included in the course in 2013, but the formulas are nonetheless included here for logical completeness. They will not be relevant to Quiz 3.)

Ideal Gas of Classical Nonrelativistic Particles:

$$n_i = g_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} .$$

$$\begin{aligned}
\text{where } n_i &= \text{number density of particle} \\
g_i &= \text{number of spin states of particle} \\
m_i &= \text{mass of particle} \\
\mu_i &= \text{chemical potential}
\end{aligned}$$

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

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