PROFESSOR: OK. In that case let's get going. In today's lecture, we're going to be sort of splitting the lecture of things, if the timing goes as I plan. We're going to start by finishing talking about the geodesic equation. And then if all goes well, we will start talking about the energy of radiation-- completely changing topics altogether.

I want to begin, as usual, by reviewing quickly what we talked about last time, just to remind us where we are. Last time, we at first, at the beginning of lecture, talked about how to add time into the Robertson-Walker Metric. And this is the formula that we claimed was the correct one.

For a spacetime metric, ds squared, the meaning is closely analogous to the meaning that it would have in special relativity. The main difference being that in special relativity we always talk about what is observed by inertial frames of reference and inertial observers.

In general relativity, the concept of an inertial observer is not so clear cut, but we can talk about observers for whom there is no forces acting on them other than possibly gravitational forces. And whether or not there are gravitational forces is always, itself, a framed dependent question. So it does not have a definite answer.

So observers for which there is no forces acting on them other than gravitational forces are called free-falling observers. And they play the role of inertial observers that the inertial observers play in special relativity. So if ds squared is positive, it's the square of the spatial separation measured by a local free-falling observer, for whom the two events happen at the same time.

Last time, I think, I did not really mention or emphasize the word local. But the point is that in general relativity we expect in any small region one can construct an accelerating coordinate system in which the effects of gravity are canceled out, as the equivalence principle tells us we can do.

And then you essentially see the effects of special relativity. But it's only a small region, in principle, on an infinitesimal region. So these measurements that correspond to special relativity measurements are always made locally by an observer who is, in principle, arbitrarily close to the events being measured.

If ds squared is negative, then it's equal to minus c squared times the square of the time separation that would be measured by a local free-falling observer for whom the two events happen at the same location. I should point out that a special case of this is an observer looking at his own wristwatch. His own wristwatch is always at the same location relevant to him, so it's a special case of this statement.

So it says that ds squared is equal to minus c squared times the time that a free-falling observer would read on his own wrist watch. And If ds squared is 0, it means that the two events can be joined by a light pulse going from one to the other.

Having said this, we can go back to this formula and understand why the formula is what it is. The spatial part is what it is because any homogeneous and isotropic spatial metric can be written in this form. And we are assuming that the universe we're describing is homogeneous and isotropic.

The dc squared piece is really dictated by item two here. We want the t that we write in this metric to be the cosmic time variable that we've been speaking about. And that means that it is the time variable measured on the watches of observers who are at rest in this coordinate system. And that means that it has to be simply minus c squared dt squared. Or else dt would not have the right relationship to a ds squared to be consistent with what the s squared is supposed to be.

And then we also talked about why there are no dt dr terms, or dt d theta, or dt d phi. We said that any such term would violate isotropy. If you had a dt dr term, for example, it would make the positive dr direction different from the negative dr direction. And that can't be something that happens in an isotropic universe. That then is our metric for cosmology, the Roberrtson-Walker Metric.

And another important thing is what is it good for, now that we decided that's the right metric? What use is to us? And what we haven't done yet, but it's actually on the homework, we need the full spacetime metric to be able to find geodesics, to be able to learn the paths of particles moving through this model universe.

So we will be making important use of this Roberrtson-Walker Metric with its spacetime contributions. OK. Any questions about that? Now I'm ready to change gears to some extent. Yes, Ani?

AUDIANCE: So in general, the spatial part of the metric, we can get from the geometry? And in general, can you just add a minus c squared dt square for the temporal part?

PROFESSOR: It's not quite general. Remember we used an argument based on isotropy here. So I think it's safe to say that any metric you'll find in this class is likely to have the time entering, and nothing more complicated than minus c squared dt squared. But it's not a general statement about general relativity. Any other questions? Yes.

AUDIANCE: [INAUDIBLE],

PROFESSOR: OK, the question is, what would be a circumstance where we would have to deal with something more complicated? The answer would be, I think, all you need is to add to this model universe perturbations that break the uniformity. If we tried to describe the real universe instead of this ideal universe, where our ideal universe is perfectly isotropic and homogeneous, if it said we wanted to describe the lumps and bumps of the real universe, then it would become more complicated. And we would probably need a dt, dr term.

OK, next we went on to talking about the geodesic equation. According to General Relativity, the trajectories of particles that have no forces acting on them other than gravity, these free-falling observers, are geodesics in the spacetime.. So that means we want to learn how to calculate geodesics, which means paths whose length is stationary under small variations.

So we considered first just simple geodesics in the spatial metric, because that's easier to think about. What is the shortest distance between two points in a space that's described by some arbitrary metric?

So first we talked about how we describe the metric. And we introduced two features in this first formula here. One is that instead of calling the coordinates XYZ or something like that. We called them x1, x2, x3, so that we could talk about them all together in one formula without writing separate pieces for the different coordinates. So i and j represent 1, 2, and 3, or just 1, and 2, which is the labeling of the spatial coordinates.

And the other important piece of notation that is introduced in that formula is the Einstein summation convention. Whenever there's an index, like i and j here, which are repeated with one index lower and one index upper, they're automatically summed over all of the values that the coordinates take, without writing summation sign.

It saves a lot of writing. And it turns out that one always sums under those circumstances, so there's no need to write the sums with the summation sign.

Next we want to ask ourselves, how are we going to describe the path? Before we can find the minimum path, we need at least a language to talk about paths. And we could describe a path going from some point A to some point B, by giving a function x supra i of lambda. Well, lambda is an arbitrary parameter, that parametrizes the path.

x supra i are a set of coordinates. i runs over the values of all the coordinates of whatever system you're dealing with. And you construct such a function where xi of 0 is the starting point, which are the coordinates of the point A. And xi of some value lambda f, where f just stands for final, will be the end of the path. And it's supposed to end at point B. So the final coordinates of the path should be x supra i sub b, the coordinates of the point B.

Then we want to use this description of the path to figure out what the length is of a segment of the path. And then the full length will be the sum of the segments. So for each segment, we just apply the metric to the change in coordinates. The change in coordinates, as lambda is varied, is just the derivative of xi with respect to lambda times the change in lambda.

And putting that in for both dxi and dxj one gets this formula, relating ds squared, the square of the length of an infinitesimal segment to d lambda squared, the square of the parameter that describes that length. Then the full length is gotten by, first of all, taking the square root of this equation to get the infinitesimal length, ds. And then taking the integral of that over the path from beginning to end.

And that, then, gives us the full length of the path, thinking of it as the sum of the length of each of infinitesimal segment. OK? Fair enough?

Now that we have this formula for the length, now we have the next challenge, which is to figure out how to calculate the path which minimizes that length. And I didn't use the word last time, but that what is called the calculus of variations. And I looked up a little bit of the history in the Wikipedia.

The calculus of variations dates back to 1696, when Johann Bernoulli invented it, applied it to the brachistochrone problem, which is the problem of finding a path for which a frictionless object will slide and get to its destination in the least possible time. And it turns out to be a cycloid, just like the cycloid that describes our closed universes, closed matter dominating the universe.

And the problem was also solved by-- Johann Bernoulli then announced this problem to the world and challenged other mathematicians to solve it. There's a famous story that Newton noticed this question in his mail when he got home at 4:00 AM, or something like that, from the mint-- he was apparently a hardworking guy-- but nonetheless when he seen this problem he couldn't go to bed. He went ahead and solved it by morning, which is a good MIT student kind of thing to do.

So the technique is to consider a small variation from whatever path you're hoping to be the minimum. And we're going to calculate the first order change in the length of the path, starting from our original path, x of lambda, to some new path, x tilde of lambda. And we parametrize the new path by writing it as the old path, plus a correction.

And I've introduced a factor, alpha, multiplying the correction, because it makes it easier to talk about derivatives. And wi of lambda is just some arbitrary deviation from the original path. But we want to always go through the same starting point to the same endpoint, because there's never going to be a minimum if we're allowed to move the endpoints.

So the endpoints are fixed. And that means that this path deviation, w super i in my notation, has to vanish at the two endpoints. So we impose these two equations on the variation wi. Then what do I do is take the derivative of the path length of the varied path, x tilde with respect to alpha, and if we had a minimum length to start with, the derivative should always vanish. That is, the minimum should always occur when alpha equals 0, if the original path of the true path, the true minimum path.

And if alpha equals 0 is the minimum, the derivative should always vanish at alpha equals 0. And vice versa. If we know that this happens for every variation wi, then we know that our path is at least an extremum, and, presumably, a minimum. And the path itself is just written by the same formulas we had before, except for x tilde instead of x itself.

And I've introduced an axillary quantity, a of lambda alpha, which is just what appears inside the square root. That just saves some writing, because it has to be written a number of times in the course of the manipulations.

So our goal now is to carry out this derivative. And the derivative acts only on the integrand, because the limits of integration do not depend on alpha. So just carry the derivative into the integrand and differentiate this square root of a of lambda, which is, itself, a product of factors that we have to use-- product rule and chain rule and various manipulations.

And after we carry out those manipulations, we end up with this expression in a straightforward way involving a few steps, which I won't show again. And the complication is that what we want to do is to figure out for what paths that expression will vanish for all wi. We want it to vanish for all possible variations of the path.

And what's complicated is that wi appears here as a multiplicative factor in the first term, but as a differentiated factor in the second term. And that makes it very hard to know, initially, when those two terms might cancel each other to give you 0, which is what we're looking for. But the brilliant trick that, I guess, Newton invented, along with Bernoulli and others, is to integrate by parts. Integration by parts, I'm sure, was not a well-known procedure at that time.

But if we integrate the second term by parts, we could remove the derivative acting on w, and arrange for w to be a multiplicative factor in both terms. And a crucial thing that makes the whole thing useful is that when you do integrate by parts, you discover that you don't get any endpoint contributions, because the endpoint contributions would be proportional to wi at the endpoints.

And remember, wi has to vanish at the endpoints, because that's the condition that we're not changing the points A and B. We're always talking about paths that have the same starting point and the same ending point.

So integrating by parts, we get this expression, where now wi multiplies everything, as just simply a multiplicative factor. To write it in this form, you had to do a little bit of juggling of indices. The other important trick in these manipulations is to juggle indices, which I'll not show you explicitly.

But the thing to remember is that these indices that are being summed over can be called anything and it's still the same sum. So when you want to get terms to cancel each other, you may have to change the names of indices to get them to just cancel identically. But that's straightforward.

So we get this expression. And now we want this expression to vanish for every possible wi of lambda. And we argued that the only way it could vanish for every possible wi of lambda is if the expression in curly brackets, itself, vanishes. Yeah, if we only know the values for some particular wi of lambda, then there are lots of ways it could vanish, because it could be positive in some places and negative in others.

But the only way it could vanish for all wi is for the quantity in curly brackets to vanish. So that gives us our final, or at least, almost final expression of the geodesic equation. And that's where we left off last time, with that equation. So note that this is just an equation that would either be obeyed or not obeyed by the function x super i of lambda. It's just a differential equation involving x super i of lambda and the metric, which we assume is given.

OK. So are there any questions about that? Everybody happy? Great. OK, now we'll continue on on the blackboard.

OK, the first thing I want to do is to simplify the equation a bit. This equation is fairly complicated, because of those square roots of A's in the denominators. The square root of A is a pretty complicated thing to start with, and the square root of A here is even differentiated, because it's got the lambda making an incredible mess, if you understand all that.

So it would be nice to simplify that. And we do have one trick which we can still do, which we haven't done yet. We originally constructed our path, xi of lambda, as a function of some arbitrary parameter, lambda. Lambda just measures arbitrary points along the path. There are many, many ways to do that, an infinite number of ways that you can do that. And this formula will work for all of them, it's completely general.

The formula, when we derived it, we didn't make any assumptions about how lambda was chosen. But we can simplify the formula by making a particular choice for lambda. And the choice that simplifies things is to choose lambda to be the arc length itself. Lambda should be the distance along the path. And then we're trying to express xi as a function of how far you've already gone.

And that has the effect, if we go back to what Ai was, A of lambda really is just the path length per lambda. So if lambda is the path length itself, A is just equal to 1. I'm trying to get a formula that shows that more clearly. Here.

If we remember that this quantity is A, this tells us that ds squared is equal to A times d lambda squared. So if ds is the same as d lambda, as you've chosen your parameter to be the path length, this formula makes it clear that that's equivalent to A equal to 1.

So going back to the formula, if A is 1, we would just drop it from both sides of the equation. And all that really matters, I should point out here, maybe, because we'll be using it later, is that A is a constant. As long as A is a constant, it will not be differentiated, and then it will cancel on the left side and the right side.

So we don't necessarily care that it is 1, but we do care that it's a constant. And then it just disappears from the formula. And then we get the simpler formula. And now we'll continue on the blackboard. The simpler formula is just dds of gij dxj ds is equal to 1/2 times the derivative of gjk, with respect to xi, times dxj ds dxk ds, where s is equal to the path length.

So I've replaced lambda by s, because we set lambda equal to s. And s has a more specific meaning than lambda did. Lambda was a completely arbitrary parametrization of the path. So this one deserves a big box, because it really is the final formula for geodesics. Once we write it in terms of different letters, we will later, but this actually is the formula.

Now I should mention just largely for the sake of your knowing what's going on, if you ever look at some other general relativity books, this is not the formula that the geodesic equation is usually written in. Frankly, it is the best form. If you want to find the geodesic, usually this form of writing the equation is the easiest.

But most general relativity books prefer instead to just give a formula for the second derivative, here. Which involves just expanding this term, and then when we shuffle things, to try to simplify the expressions. So one can write, to start, d ds of gij dxj ds.

We're just going to expand it.

Now we're going to be making use of all the rules of calculus that we've learn. Every rule you've ever learned will probably get used in this calculation. So it will be using product rule, of course, because we have a product of two things here. But we also have the complication that gij is not explicitly a function of s. But gij is a function of position. And the position that one is that for any given value of s depends on s, because we're moving along the path, x super i of s.

So the gij here, is evaluated at x super i of s. I should give this a new letter. x super k of s. So it depends on s, through the argument of its argument. So that's a chain rule situation.

And what we get here is, from just differentiating the second factor, that's easy. We get gij DC d squared xj ds squared. And then, from the derivative of the derivative chain rule piece, we get the partial of gij, with respect to xk times the dxj ds times dxk ds.

And then to continue, this piece gets brought over to the other side, because we're trying to get an equation just for the second derivative of the path. So then we get g sub ij d squared x super j ds squared is equal to 1/2 di-- I'll define that in a second-- g sub jk minus 2 dk gij dxi ds dxj ds. where this partial derivative with the subscript is just an abbreviation for the derivative with respect to the coordinate with that index. So that's just an abbreviation.

Now you could think of this as a matrix times a vector is equal to an expression. What we like to do is just get an expression for this vector. So if we think of it as a matrix times a vector, all we have to do is invert the matrix to be able to get an expression for the vector itself.

Yes!

AUDIANCE: Should that closing parenthesis be more [INAUDIBLE]?

PROFESSOR: Oh, Yeah, I think you're right, it doesn't look right. Yeah. Thank you This has to multiply everything. Oops! OK, OK. Given enough chances I'll get it right. OK, now everybody happy this time? Thank you very much for getting it straight.

OK, So as I was saying, we want to isolate this second derivative. We're hoping to get just an expression for the second derivative. And this can be interpreted as a matrix times a vector equals something. We want to just invert that matrix. Yes?

AUDIANCE: Isn't the ds and [? the idx ?] [INAUDIBLE]?

PROFESSOR: Oh, do I have that wrong too, perhaps? I think we want j and k there, that don't we? OK, attempt number four, or did I lose count as well? j and k are the indices and the i matches the free i on the left. And all the other indices are sound. I think, probably, I finally achieved the right formula. Thanks for all the help.

So inverting a matrix, the principal is a straightforward mathematical operation. In general relativity, we give a name to the inverse metric, and it's the same letter g with indices, with superscripts instead of subscripts. And that's defined to be the matrix inverse. So g super ij is defined to be the matrix inverse of g sub ij.

And to put that into an equation, we could say that if we take g with upper indices-- and I'll write those upper indices as i and l-- and multiply it by a g with lower indices l and j, when you sum over adjacent indices in this index notation, that's exactly what corresponds to the definition of matrix multiplication.

So this is just the matrix g with upper indices times the matrix g with lower indices, and it's the i j'th element of that matrix. And we're saying it should be the identity matrix, and that means that the i j'th element should be 0 if it's off diagonal, and 1 if it's diagonal, if i equals j.

And that's exactly the definition of a chronic or a delta. So this is equal to delta ij. We remember that delta ij is 0 if i is not equal to j, 1 if i is equal to j. That's the definition. And it corresponds to that identity matrix in matrix language. So this is the relationship that actually defines g super il. And it is just the statement that g with upper indices is the matrix inverse of g with lower indices.

Using this, we can bring this g to the other side essentially by multiplying by g inverse. And I will save a little time by not writing that out in gory detail, but rather I'll just write the result. And the result is written in terms of a new symbol that gets defined, which is an absolutely standard symbol in General Relativity.

The formula is d squared x i, ds squared is equal to-- we know it's going to be equal to stuff times the product of two derivatives. And the stuff that appears is just given a name, capital gamma, which has an upper index i, which matches the left hand side of the equation. And two lower indices, which I'm calling j and k, which will get summed with the derivatives that follow, d x j ds, dx k, ds.

And this quantity, gamma super i sub jk are just the terms that would appear when we do these manipulations. And I'll write what they are. Gamma super i sub jk is equal to 1/2 g super il times the derivative with respect to j of g sub lk, plus the derivative with respect to k of g sub lj. And then minus the derivative with respect to l of g sub jk. And this quantity has several different names.

Everybody agrees how to define it up to the sign. There are different sign conventions that are used in different books. And there are also different names for it. It's often called the affine connection. If you look, for example in Steve Weinberg's General Relativity book, he calls it the affine connection.

It's also very often called the Christofel connection, or the Christofel symbol. And frankly those are the only names for it that I've seen, personally. But there's a book about [INAUDIBLE] by Sean Carroll which is a very good book. And he claims that it's sometimes also called the Riemann connection And it's also sometimes called the Levi-Civita connection. So it's got lots of names, which I guess means lots of people's independently invented it.

But in any case, that's the answer. And it's just a way of rewriting the formula we have up there. And for solving problems, the formula, the way we wrote up there, is almost always the best way to do it. So this is really just window dressing, largely for the purpose of making contact with other books that you might come across.

OK, so that finishes the derivation of the geodesic equation. Now I'd like to give an example of its use. But before I do that, let me just pause to ask if there are any questions about the derivation? OK.

So on your homework, you will, in fact, be applying this formalism to the Robertson-Walker metric. And you'll learn how moving particles slow down as they move through an expanding universe, completely in an analogy to the way photons, which we've already learned, lose energy as they travel through an expanding universe. So particles with mass also lose energy in a well-defined way, which you'll be calculating on the homework.

For example, though, I'll do something different. A fun metric to talk about is the Schwarzschild metric, which describes, among other things, black holes. It in principle, describes anything which is spherically symmetric and has a gravitational field. But black holes are the most interesting example, because it's where the most surprises lie.

So the Schwarzschild metric has the form ds squared is equal to minus c squared d tau squared, which is equal to-- this is just a definition, it defines d tau-- but in terms of the coordinates, it's minus 1 minus 2 G, Newton's constant, M, the mass of the object we're discussing-- the mass of the black hole, if it is a black hole-- divided by r times c squared, r is the radial coordinate, times c squared dt squared, plus 1 minus 2 GM over rc squared times dr squared plus r squared times d theta squared plus sine squared theta d phi squared.

Now here, theta and phi are the usual polar angles. We're using a polar coordinate system. So as usual, theta lies between 0 and pi. 0, what we might call the North Pole, and pi what we might call the South Pole. And phi is what is often called an azimuthal angle, it goes around. And the way one describes coordinates on the surface of the Earth, phi would be the longitude variable.

So 0 is less than or equal to phi is less than or equal to 2 pi where phi equals 2 pi is identified with phi equals 0. And you can go around and come back to where you started.

Now notice that if we set capital M, the mass of this object equal to 0, the metric becomes the trivial metric of Special Relativity written in spherical polar coordinates. So all complications go away if there's no mass. The object disappears. But as long as the mass is non-zero there are factors that multiply the dr squared term and the c squared dt squared term.

Notice that the factors that do that multiplying-- now one of these should be inverted. Important inverse, it's a minus 1 power for that factor. Notice that r can be small enough so that these factors will vanish. And the place where that happens is called the Schwarzschild radius after the same person who invented the metric.

So r sub Schwarzschild is equal to 2 GM divided by c squared. When little r is equal to that, this quantity in parentheses vanishes, which means we get infinity here, because it's inverted, and we get a 0 there.

Now when a term in the metric is either 0 or infinite, one calls that a singularity. In this case, it's a removable singularity. That is, the Schwarzschild singularity is only there because Schwarzschild chose to use these particular coordinates. These are simpler than other coordinates. He wasn't foolish to use them.

But the appearance of that singularity is really caused solely by the choice of coordinates. There really is no singularity at the Schwarzschild horizon.

And that was shown some years later by other people constructing other coordinate systems. The coordinate system is best known today that avoids the Schwarzschild singularity is a coordinate system called the Kruskal coordinate system. But we will not be looking at the Kruskal coordinate system in this class. Leave that for the GR class that you'll take some time.

OK, now the masses sum parameter, notice that the metric is completely determined by the mass. And that's the same situation as we found in Newtonian gravity. The metric outside of the spherically symmetric object, by the gravitational field in Newtonian Physics outside of a spherical symmetric object, depends only on the total mass, which does not depend, at all, on how it's distributed as long as it's spherically symmetric.

And the same thing here. As long as an object is spherically symmetric, the gravitational field outside of the object will always look like that formula. Now there are still two cases-- outside of the object could be larger than or smaller than this Schwarzschild radius.

So for an object like the sun, the Schwarzschild radius, we could calculate it-- and it's calculated in the notes-- it's about two or three kilometers. Hold on and I'll tell you more accurately. It's 2.95 kilometers, the Schwarzschild radius of the sun. But the sun, of course, is much bigger than that.

And that means that the sun doesn't have a Schwarzschild horizon. That is, at 2.95 kilometers from the center of the sun there's still sun. It's not outside the sun. This metric only holds outside the spherically symmetric object.

So it does not hold inside the sun. The place where this has the apparent singularity the metric is not valid at all. So there is nothing that even comes close to anything worth talking about, as far as the Schwarzschild singularity for an object like the sun. But if the sun were compressed to a size smaller than 2.95 kilometers with the same mass, then these factors would be relevant at the places where they vanish. And whatever consequences they have, we would be dealing with.

Even though r equals r Schwarzschild is not a singular point, it is still a special point. What you can show-- we won't-- but what we can show is that that is the horizon. Meaning that if an object falls inside this Schwarzschild radius, there is no trajectory that will ever get it out. Yes?

AUDIANCE: Say a star is just incredibly dense at its core. Is it possible to have suppression of some fractional life of a star that's from that mass that it's contained? Or like a fusion reaction that is going on with the net radius?

PROFESSOR: OK, could there be a horizon inside of a star? I think is what you're asking, basically.

AUDIANCE: One that actually affects the--

PROFESSOR: One that really is a horizon.

AUDIANCE: That's outside.

PROFESSOR: Right. If this were the sun you were describing, this formula would just not be valid inside. There would be no horizon inside. But you're asking a real valid question. If a star had, for some reason, a very dense spot in the middle, could it actually form a horizon inside the material?

And the answer is, yes, it could. It would not be stable. The material would ultimately fall in, but it could happen. Yes?

AUDIANCE: So like our galaxy has a super massive black hole in the center.

PROFESSOR: That's right. Our galaxy does have a super massive black hole in the center.

AUDIANCE: Yeah. So you can consider that as like a larger mass that has black hole, area?

PROFESSOR: Right! Right! That's right. The comment is that if we go from a star to something bigger than a star we have perfectly good example in our own galaxy, where there is a black hole in the center, but there is still mass that continues outside of that. And the black hole is accreting, more matter does keep falling in, it's not really stable. But it certainly does exist, and can exist. Any other questions?

Well, our goal now is to calculate a geodesic. And I will just calculate one geodesic. I will calculate what happens if an object starts at some fixed radius at rest and is released and falls into this black hole.

I first want to just rewrite the geodesic equation in terms of variables that are more appropriate for this case. When I wrote that, I had a mind just calculating the geodesics in space, looking for the shortest path between two points.

The geodesic that we're talking about when we're talking about an object in general relativity moving along the geodesic is a geodesic that's a time-like geodesic. That is, any increment along the geodesic is a time-like interval, or following a particle. Particles travel on time-like trajectories in relativity.

So the usual notation for time is something like tau rather than s, which is why I wrote it this way. ds squared is just defined to be minus c squared d tau squared. So d tau squared has no more or less information than ds squared, but it has the opposite sign and a difference by a factor of c squared, as well.

And another change in notation which is a rather universal convention is that, when we talk about space alone we use Latin indices, ijk.. When we talk about spacetime, where one of the indices might be 0 referring to the time direction, then we usually use Greek indices, mu, nu, lambda.

So I'm going to rewrite the geodesic equation using tau as my parameter instead of s, since we're talking about proper time along the trajectory instead of distances. And using Greek letters instead of Latin letters, because we're talking about spacetime rather than just space.

So otherwise what I'm going to write is just identical to that. So really is nothing more than a change in notation. d d tau of g mu nu, dx super nu d tau. And it is equal to 1/2 times the partial of g lambda sigma with respect to x nu dx lambda d tau dx sigma d tau.

Now you might want to go through the calculation and make sure of the fact that now we're dealing with a metric which is not positive, definite, doesn't make any difference. But it doesn't. It does mean that now we certainly have possibilities of getting maxima and stationary points as well as minima, because of the variety of signs that appear in the metric.

But otherwise, the calculations of the geodesic equation goes through exactly as we calculated it. And the only thing I'm doing here, relative to what we have there, is just changing the notation a bit to conform to the notaion that is usually used for talking about spacetime trajectories.

Since we're talking about radio trajectories, we're just going to release a particle at rest and then it will fall straight towards the center of our spherical object, we know by symmetry that it's not going to be deflected in the positive theta or the negative theta, or the positive phi or negative phi directions, because that would violate isotropy. It would violate the rotational symmetry that we know as part of this metric. This Is just the metric of the surface of the sphere.

So theta and phi will just stay whatever values they have when you drop this object. So we will not even talk about theta and phi. We will only talk about r and t, how particle falls in as a function of time.

And then it turns out to be useful to just first write down what the metric itself tells us. And we'll divide by d tau. So we could talk about derivatives with respect to tau.

So changing an overall sign, since everything's going to be negative and we'd rather have everything be positive, we can just rewrite the metric equation as saying, that c squared is equal to 1 minus 2 GM over rc squared, times c squared times dt d tau squared minus 1 minus 2 GM over rc squared inverse times dr d tau squared.

So this is nothing more than rewriting this equation saying d theta is equal to 0 and d phi will be 0. Written this way, though, it tells us that we can find dt d tau, for example, if we know dr d tau. And we also know where we are, you know, little r. And we'll be using that, shortly.

To continue a little further, we're going to introduce some abbreviations just so we're don't have to write so much. I'm going to define little h of r as just one minus r Schwarzschild over r. And this is also 1 minus 2 GM over rc squared. That's a factor that keeps recurring in our expression for the metric. Yes?

AUDIANCE: The second to last equation is supposed to be a c squared in between the two parenthesis?

PROFESSOR: Probably. Yes, thank you. G squared, right? Thanks a lot. In terms of h of r, we can rewrite that equation slightly more simply. I'm going to bring things to the other side and write it as c squared times dt d tau squared is equal to c squared h inverse of r plus h to the minus 2 of r times dr d tau squared.

This is just a rewriting of the above equation, making use of the new notation that we've introduced. And this is the form we will be using. It explicitly tells us how to find dt d tau in terms of other things. So dt d tau is not independent.

Since we know dt d tau in terms of dr d tau. If We get an expression for dr d tau we're sort of finished. We could find everything we want to know about t from the equation we just wrote. So it turns out that all we need to do to calculate this radial trajectory is to look at the component of the metric where that free index, mu, mu is the index that's not summed, we're going to set mu equal to r.

Remember mu is a number that corresponds to a coordinate. And we're going to set mu equal to the value that corresponds to the r coordinate. And that will be sufficient to get us our answer.

When we do that, the equation becomes d d tau of g sub r. Now the second index, nu in the original expression, is summed from 0 to 3 for the gr case, where we have four coordinates, one time and three spatial coordinates, but we only need to write the terms where gr nu variable is non-zero. And the metric itself is diagonal.

So if one index is a little r, the other index has to also be r, or else it vanishes. So the only value of nu that contributes to the sum is when nu is also equal to the r coordinate. So we get g sub rr d xr-- which, in fact I'll write it as just dr. x super r is just the r coordinate, which we also just call r times d tau is equal to 1/2 dr.

And now, on the right-hand side, we're summing over lambda and sigma. And lambda and sigma have to have the property that g sub lambda sigma depends on r, or else the first factor will vanish. And furthermore, g sub lambda sigma has be non-zero, for the values of lambda and sigma that you want, which means that lambda and sigma for this case has to be equal to each other, because we have no off-diagonal terms to our metric.

So the only contributions we get are from g sub rr and g sub tt. So you get the derivative with respect to r of g sub rr times dr d tau squared. This become squared, because lambda is equal to sigma. And then plus 1/2 drg sub tt times dt d tau squared.

And note that buried in here is, if we expand this, the second derivative of r with respect to time-- respect to tau. So we can extract that and solve for it. And things like dt d tau will appear in our answer, initially, because it's here already.

But we could replace dt d tau by this top equation and eliminate it from our results. And I'm going to skip the algebra, which is straightforward, although tedious. I urge you to go through it in the notes. But the end result ends up being remarkably simple, after a number of cancellations that look like surprises.

And what you find in the end-- and it's just the simplification of this formula, nothing more-- you find that d squared r d tau squared is just equal to minus Newton's constant times the mass divided by r squared.

Now this is rather shocking, and even looks exactly like Newtonian mechanics. However, even though it looks like Newtonian mechanics, it's not really the same as Newtonian mechanics, because the variables don't mean quite the same thing.

First of all, even r does not really mean radius in the same sense as radius is defined by Newton. In Newtonian mechanics, radius is the distance from the origin. If we wanted to know the distance from the origin, we would have to integrate this metric. And in fact, there isn't even an actual origin here, because you would have to go through the singularity before you get there. And you really can't. That integral is not really even defined.

Although, of course, if we had something like the sun, where the metric was different from this small r, then we could integrate from r equals 0, and that would define the true radius, distance from the center. But it would not be r. It would be what you got by integrating with the metric. So r has a different interpretation than it does for Newtonian physics.

I might add, it still has a simple interpretation. If you look at this metric, the tangential part, the angular part, is exactly what you have for Euclidean geometry. It's just r squared times the same combination of d theta and d phi as appears on the surface of a sphere.

So little r is sometimes called the circumferential radius, because it really does give you the circumference of circles at that radial coordinate. If we went around in a circle at a fixed r, the circle would involve varying phi, for example, over a range of 2 pi, we really would see a total circumference of 2 pi little r. So r is related to circumferences in exactly the way as it is in Euclidean geometry. But it's not related to the distance from the origin in the same way as it is in Euclidean geometry.

In addition, tau, here, is not the universal time that Newton imagined. But rather, tau is measured along the geodesic. It is just ds squared, but remember, ds squared is being measured along the geodesic, which means that it is, in fact, the proper time as it would be measured by the person falling with the object towards the black hole. So tau is proper time as measured by the falling object. And that follows from what we know about the meaning of the metric itself.

OK, that said we would now like to just study this equation more carefully. And since the equation itself still has the same form as what you get from Newton, if you remember what you would have done if this was 801, you can, in fact, do exactly the same thing here.

And what you probably would have done, if this was 801, would be to recognize that this equation can be integrated. We can write the equation as d d tau of 1/2 dr d tau squared minus GM/r equals 0. I you carry out these derivatives you would get that equation. And this is just the conservation of energy version of the force equation.

And that tells us that this quantity is a constant. If we drop the object from some initial position, r sub 0, and we drop it with no initial velocity, we just let go of it at r sub 0, that tells us what this quantity is when we drop it. It's minus GM over r sub 0. This piece vanishes if there is no initial velocity. And that means it will always have that value.

And knowing that, we can write dr d tau is equal to-- just solving for that-- minus the square root of 2GM times r0 minus r over r r0 I've collected two terms and put them over a common denominator and added them. So this is not quite as obvious as it might be. But this is just the statement that that quantity has the same value as it did when you started.

Now this can be further integrated. We can write it as dr over-- bringing all this to the other side-- is equal to d tau. And then integrate both sides. Notice when I bring this to the other side and bring the d tau to the right., everything on the left-hand side now only depends on r. So this is just an explicit integral over r that we can do.

And I will just tell you that when the integral is done we get a formula for tau as a function of r. And it's equal to the square root of r sub 0 over 2GM times r0 times the inverse tangent of the square root of r0 minus r over r plus the square root of r times r0 minus r.

So when r equals r0, this gives us 0, and that's what we want. When we start we're at r0, or time 0, or proper time 0. And then as r gets smaller, as it falls in, time grows. And this gives us the time as a function of r. We might prefer to have r as a function of time, but that formula can't really be inverted analytically. So that's the best we can do.

Now one thing that you notice from this is that nothing special happens as r decreases all the way to 0. Even when you plug in r equals 0 here, you just get some finite number. So in a finite amount of time, the observer would find himself falling through the Schwarzchild horizon and all the way to r equals 0.

I didn't mention it but r equals 0 is a true singularity. Our metric is also singular when r equals 0. These quantities all become infinite. And physically what would happen is that, as the object falling in approaches r equals 0, the tidal forces, that is the difference in the gravitational force on one part of the object verses another, will get stronger and stronger. And objects will just be ripped apart.

And the ripping apart occurs as being spaghetti-ized, that is, the force on the front gets to be very strong compared to the force on the back. So I'll just get stretched out along the direction of motion.

Now the curious thing is what this looks like if we think of it not as a function of the proper time measured by the wrist watch of the object falling in but rather, we could try to describe it in terms of our external time variable. The variable t that appears in the Schwarzchild metric.

And to do that, to make the conversion, we want to calculate what the dr dt is, instead of dr d tau. Like maybe an analogous formula, in terms of t. And to get that, we use simply chain rule here. dr dt is equal to dr d tau-- which we've already calculated-- times d tau dt. And d tau dt is 1 over dt d tau. If you just have two variables that depend on each other. The derivatives are just the inverse of each other.

So this could be written as dr d tau-- which we've calculated-- divided by dt d tau. And dt d tau we've really already calculated as well, because it's just given by this formula here. So we could write out what that is and figure out how it's going to behave as the object approaches the Schwarzchild radius.

So it becomes dr dt is equal to, I'll just write the numerator as dr d tau given by that expression. But what's behaving in a more peculiar way is the denominator, which is h inverse of r plus c to the minus 2, h to the minus 2 of r times dr d tau squared.

So now we want to look at this function h inverse of r. And this just means 1/h of r. It doesn't mean functional inverse. That is just equal to r over r minus r Schwarzchild.

And we're going to be interested in what happens when r gets to be very near r Schwarzchild, because that's where the interesting things happen, as you're approaching the Schwarzchild horizon. And that means that the behavior of the numerator won't be important. The denominator will be going up, and that's what will control everything.

So we can approximate this as just r Schwarzchild over r minus r Schwarzchild. And this is for r near r Schwarzchild. We've replaced the numerator by a constant.

And then if we look at this formula, this is going to blow up as we approach the horizon. This is the square of that quantity. It will blow up faster than the first power of that quantity. And therefore, this will dominate, the denominator of the expression. We can ignore this.

When this dominates, the dr d tau pieces cancel. So that's nice. We don't even need to think about what the dr d tau is. And what we get near the horizon is simply a factor of c times r minus r Schwarzchild over r Schwarzchild. It's basically just h. This becomes upstairs with a plus sign. And the square root turns it into h instead of h squared. So this is the inverse of that.

OK, now if we try to play the same game here as we did here, to determine what our time variable behaves as a function of r, instead of the proper time variable tau, what we find is that t of r-- this is for r near r Schwarzchild-- is about equal to minus r sub s over c times the integral up to r of dr prime over r prime minus rs.

This is dr dt. Yeah, this was dr dt from the beginning. I forgot to write the r somehow.

AUDIANCE: Doesn't that [INAUDIBLE]?

PROFESSOR: Yeah, I didn't write the lower limit of integration. I was about to comment on that. The integrand that we're writing is only a good approximation whenever we're near r. So whatever happens near the lower limit of integration, we just haven't done accurately.

So I'm going to just not write a lower limit of integration here, meaning that we're interested only in what happens as the upper limit of integration r becomes very near r Schwarzchild. And everything will be dominated by what happens near the upper limit of integration.

AUDIANCE: So would you just integrate over on [INAUDIBLE] for that?

PROFESSOR: That's right, that's right. We just integrated over a small region near, r Schwarzchild. Nu r, which is also about equal to r Schwarzchild. And the point is, that this diverges logarithmically as r approaches r Schwarzchild. So it behaves approximately as minus r Schwarzchild over c times the logarithm of r minus r Schwarzchild.

So as r approaches r Schwarzchild, this quantity that's the argument of the logarithm gets closer and closer to 0. It gets smaller and smaller approaching 0. But the logarithm of a very small number is a negative number, a large negative number. And then there's a minus sign here. You get a large positive number and it diverges.

As r approaches r Schwarzchild the time variable approaches infinity. And that means that at no finite time does the object ever reach the Schwarzchild horizon. But as seen from the outside, it takes an infinite amount of time for the object to reach the Schwarzchild horizon. As time gets larger and larger, the object gets and closer to the Schwarzchild horizon, asymptotically approaching it but never reaching it.

So this, of course, is very peculiar, because from the point of view of the person falling into the black hole, all this just happens in a finite amount of time and is over with. From the outside, it looks like it takes an infinite amount of time. And weird things like this can happen because of the fact that in general relativity time is a locally measured variable. You measure your time, I measure my time. They don't have to agree. And in this case, they can disagree by an infinite amount, which is rather bizarre, but that's what happens.

So according to classical general relativity, when an object falls into a black hole, from the point of view of the object nothing special would happen as that object crossed the Schwarzchild horizon. Everybody believed that that was really the case until maybe a couple years ago.

Now it's controversial, actually. At the classical level, everybody believes that's still true. I mean, classical general relativity says that an object can fall through the Schwarzchild horizon and then nothing happens. It's not really a singularity.

But the issue is that when one incorporates, or attempts to incorporate, the effects of quantum theory, which nobody really knows how to do in a totally reliable way, then there are indications that there's something dramatic happening at the

Schwarzchild horizon. The phrase that's often used for what people think might be happening at the horizon is the word firewall.

So whether or not there is a firewall at the horizon, is not settled at this point. Certainly, though, classical general relativity does not predict the firewall. If it exists, all the arguments that say it might exist are based on the quantum physics of black holes, and black hole evaporation, and things like that.

As you know quantum mechanically, the black holes are not stable, either, if they evaporate-- as was derived by Stephen Hawking in, I think, 1974. But that's strictly a quantum effect. It would go to 0 as h bar goes to 0, and, at the moment, we're only talking about classical general relativity. So the black hole that we're describing is perfectly stable. And nothing happens if you fall through the horizon. Except from the outside, it looks like it would take an infinite amount of time just to reach the horizon.

So we'll stop there. I guess I'm not going to get to talk about the energy associated with radiation. But we'll get to that on Thursday. So see you folks on Thursday.