8.311: Electromagnetic Theory   Problem Set # 2   Due: **2/19/04**

Conservation laws. Magnetic dipole.

Reading: Schwinger, Chap. 3, 4, and Chap. 28 (or Jackson, Chap. 5, 6)

1. Angular momentum conservation, Schwinger, Prob. 3.5

Show that the angular momentum conservation law for the electromagnetic field can be written as

$$\frac{\partial}{\partial t}\mathcal{T} + \nabla \cdot \mathcal{K} + \mathbf{r} \times \mathbf{f} = 0 \tag{1}$$

where $\mathbf{f}$ is the Lorentz force. Here the angular momentum density is $\mathcal{T} = \mathbf{r} \times \mathbf{G}$ and the angular momentum flux tensor is defined in terms of Maxwell stress tensor T_{ij} as

$$\mathcal{K} = -\mathbf{T} \times \mathbf{r}$$

where the cross product refers to the second index of T_{ij}.

2. Schwinger, Probs. 2.1, 2.2

a) Write Maxwell's equations with magnetic charge

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{j}_e\,, \quad \nabla \cdot \mathbf{B} = 4\pi\rho_m \tag{2}$$

$$-\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \frac{4\pi}{c}\mathbf{j}_m\,, \quad \nabla \cdot \mathbf{E} = 4\pi\rho_e \tag{3}$$

in terms of a complex vector filed $\mathbf{F} = \mathbf{E} + i\mathbf{B}$, and related combinations of charge and current. Verify that the equations retain their form under the transformation

$$\mathbf{F} \to e^{-i\phi}\mathbf{F}$$

where ϕ is an arbitrary constant. Express this as a transformation of $\mathbf{E}$, $\mathbf{B}$ and the charge-current quantities. What is the geometric interpretation? What is the particular form of this transformation when $\phi = \pi/2$?

b) Suppose every charged particle carried electric *and* magnetic charge in the universal ratio $g_k/e_k = \lambda$. Is there another way of looking at this situation in which we would be unaware of magnetic charges?

3. Schwinger, Prob. 3.7

As in Problem 2, let $\mathbf{F} = \mathbf{E} + i\mathbf{B}$, $\mathbf{F}^* = \mathbf{E} - i\mathbf{B}$. Identify the scalar $\frac{1}{8\pi}\mathbf{F}\cdot\mathbf{F}^*$ the vector $\frac{1}{8\pi}\mathbf{F}\times\mathbf{F}^*$ and the tensor

$$\frac{1}{8\pi}\left(\mathbf{F}\mathbf{F}^* + \mathbf{F}^*\mathbf{F}\right)_{ij} = \frac{1}{8\pi}\left(F_i F_j^* + F_i^* F_j\right)$$

What happens to these quantities if $\mathbf{F}$ is replaced by $e^{-i\phi}\mathbf{F}$?

4. Magnetic dipole (force and torque)

a) Consider a wire loop of an arbitrary shape, carrying current I and placed in a uniform external magnetic field $\mathbf{B}$. Find the total force and torque on the loop. Express the answer through the magnetic dipole $\mathbf{m} = \frac{1}{2c} I \oint \mathbf{r} \times d\mathbf{l}$ of the loop.

b) Consider the loop of part a) in a weakly nonuniform field

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla \mathbf{B}|_{\mathbf{r}=\mathbf{r}_0} + O((\mathbf{r} - \mathbf{r}_0)^2)$$

where $\mathbf{r}_0$ is chosen near the loop center. Find the total force on the loop, and express it through the loop dipole moment $\mathbf{m}$.

5. Magnetic dipole (field and interaction)

a) Show that the magnetic field of the current loop of Problem 4 at distances much larger than the loop size is given by the magnetic dipole formula

$$\mathbf{B}(\mathbf{r}) = \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{|\mathbf{r}|^3} \tag{4}$$

b) Using the result of part a) or otherwise, find the potential energy of interaction between two magnetic dipoles $\mathbf{m}_1$, $\mathbf{m}_2$ located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

6. Rotating sphere

A spherical shell of radius a carries charge q which is distributed uniformly over the surface. The sphere is rotating about the z axis with an angular velocity ω.
a) Find the current density $\mathbf{j}$ and the magnetic moment $\mathbf{m}$ of the sphere.
b) Write down the equations for the magnetic field $\mathbf{B}$ inside and outside the sphere and the conditions at the boundary $|\mathbf{r}| = a$ relating the field in the inner and outer regions. Find the magnetic field in the entire space. (Hint: Assume that the field is uniform inside and of a dipole form (4) outside and match the inner and outer field values at the boundary)
c) Relate the field outside the sphere to the sphere magnetic dipole moment $\mathbf{m}$.
d) Find the electromagnetic angular momentum of the system.

7. (Optional problem) Electric and magnetic charge system, Schwinger, Prob. 3.8

a) Electric charge e is located at the fixed point $\frac{1}{2}\mathbf{R}$. Magnetic charge g is stationed at the fixed point $-\frac{1}{2}\mathbf{R}$. Write down the momentum density $\mathbf{G}$ at an arbitrary point $\mathbf{r}$. Verify that it is divergenceless by writing it a a curl.
b) Evaluate the electromagnetic angular momentum $\mathcal{T}_{total} = \int \mathbf{r} \times \mathbf{G}\, d^3r$. Recognize that it is a gradient with respect to $\mathbf{R}$. Continue the evaluation to discover that it depends only on the direction of $\mathbf{R}$, not its magnitude. This is the naive, semiclassical basis for the charge quantization condition of Dirac, $eg = \frac{n}{2}\hbar c$.