8.311: Electromagnetic Theory Problem Set # 6 Due: **3/31/04**

Lagrangian and Hamiltonian dynamics review.

Reading: Schwinger, Sec. 6.1 & Chap. 8 (or Jackson, Chap. 12)

1. Lorentz force. Starting from the nonrelativistic Lagrangian

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 + \frac{e}{c}\dot{\mathbf{r}} \cdot \mathbf{A} - e\Phi \tag{1}$$

show that the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$ give the Lorentz force equation. (Hint: Use the convective derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ to express the field change seen by moving particle.)

2. Gauge invariance. Find how the nonrelativistic Lagrangian changes under a gauge transformation of the potentials $\mathbf{A}$ and Φ. Show that the action $S \int_{t_1}^{t_2} L dt$ is not gauge invariant. However, show that the change of the action depends only on the initial and final positions $q_i(t_{1,2})$, and is independent of the path.

3. Canonical action. Consider the canonical action integral:

$$S = \int\limits_{t_1}^{t_2} \left(\sum_{i=1,\ldots,n} p_i \dot{q}_i - H(p,q,t) \right) \, dt \tag{2}$$

Show that the least action path with fixed ends $q_i(t_{1,2}) = q_i^{(1,2)}$ corresponds to $p_i(t)$, $q_i(t)$ satisfying the canonical equations of motion.

Assume that $p_i(t)$ and $q_i(t)$ (and hence the variations δp_i and δq_i) are independent functions, constrained only by $\delta q_i(t_{1,2}) = 0$.

4. Free relativistic particle. For a free relativistic particle of rest mass m, the energy is

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{3}$$

Use this as the Hamiltonian H, and from the Lagrangian

$$L = \mathbf{p} \cdot \frac{d\mathbf{r}}{dt} - H \tag{4}$$

determine the relationship between the velocity $\mathbf{v} = d\mathbf{r}/dt$ and the momentum. Compute the energy in terms of the velocity. Write the Lagrangian in terms of $\mathbf{v}$.

5. Nonrelativistic Kepler problem. Consider the nonrelativistic Kepler problem: a point charge e of mass m moving in the field of a static point charge $-e$. Write down the Lagrangian in cylindrical coordinates:

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + \frac{e^2}{\rho} \ ,$$

where it is assumed that the trajectory is in the $z = 0$ plane. Show that the angular momentum $M = \partial L/\partial\dot{\varphi}$ and the energy $E = \dot{\rho}\partial L/\partial\dot{\rho} + \dot{\varphi}M - L$ are conserved. From that, derive a first order differential equation for $\rho(t)$.

a) Consider circular trajectories. Derive the Kepler's radius-period relationship.

b) Show that a generic trajectory is a closed curve: the frequencies of radial and angular motion are equal. Show that the trajectory shape is an ellipse, with one of the two focal points at the center of attraction. (In polar coordinates, an ellipse is given by the equation $r(\theta) = r_0/(1 - \epsilon\cos\theta)$.)