8.311

PS #12 Solution

Problem 1 a) Electron driven by $E(t) = E_0(\cos\omega t, \sin\omega t)$

$m a + kx = eE \to \vec{a} = a_0(\cos\omega t, \sin\omega t)$, $a_0 = \frac{eE_0}{m(1-\omega_0^2/\omega^2)}$

Radiated power $P = \frac{2}{3}\frac{e^2}{c^3}a_0^2 = \frac{2}{3}\frac{e^4E_0^2}{m^2c^3}\left(1-\frac{\omega_0^2}{\omega^2}\right)^{-2}$

Energy flux $S = \frac{c}{4\pi}|E\times B| = \frac{c}{4\pi}E_0^2$

crosssection $\sigma = P/S = \frac{8\pi}{3}\frac{e^4}{m^2c^4}\left(1-\frac{\omega_0^2}{\omega^2}\right)^{-2}$

b) Polarizability of a sphere:

$d = \alpha E \qquad \alpha = \frac{\varepsilon-1}{\varepsilon+2}r^3$

$P_{rad} = \frac{2}{3c^3}(\ddot{d})^2 \xrightarrow{t\text{-averaging}} \bar{P} = \frac{2\omega^4}{3c^3}\alpha^2\frac{1}{2}E_0^2$

$\bar{S} = \frac{c}{4\pi}\overline{E\times B} = \frac{c}{8\pi}E_0^2 \to \sigma = \bar{P}/\bar{S} = \frac{8\pi\omega^4}{3c^4}\alpha^2$

Problem 2

a) Impulsive radiation $\frac{d^2\mathcal{E}}{do\,d\omega} = \frac{e^2}{4\pi^2c^3}\left|\frac{v_1\times n}{1-\frac{1}{c}v_1\cdot n} - \frac{v_2\times n}{1-\frac{1}{c}v_2\cdot n}\right|^2$

For $v_1, v_2 \ll c$, $\frac{d^2\mathcal{E}}{do\,d\omega} = \frac{e^2}{4\pi^2c^3}|(v_1-v_2)\times n|^2$

b) $\frac{d^2\mathcal{E}}{do\,d\omega} = \frac{1}{4\pi^2c^3}\times|e\,v\times n + e'\,v'\times n|^2 = \frac{e^2v^2}{\pi^2c^3}\sin^2\theta$

image

$v' = -v$

$e' = -e$

half-space

Problem 3 a) $\frac{d^2E}{d\Omega\, d\omega} = \frac{(Z/2)^2}{4\pi^2 c^3} \left| \frac{v\times n}{1-\frac{1}{c}v\cdot n} - \frac{v\times n}{1+\frac{1}{c}v\cdot n} \right|^2$

Note cancellation at $v \ll c$ (zero net dipole)

b) $$\frac{d\mathcal{E}}{d\omega} = \frac{(Z/2)^2}{4\pi^2 c^3}\, 2\pi \int_{-1}^{1} \frac{(v\times n)^2 \frac{1}{c^2}(v\cdot n)^2}{1-\frac{1}{c^2}(v\cdot n)^2}\, d\cos\theta$$

$$\int_{\dots} = \frac{v^4}{c^2}\int_{-1}^{1} \frac{\sin^2\theta\cos^2\theta\, d\cos\theta}{1-\left(\frac{v}{c}\right)^2\cos^2\theta} = \frac{v^4}{c^2}\int_{-1}^{1}\frac{(1-x^2)x^2\,dx}{1-\frac{v^2}{c^2}x^2}$$

$$\frac{(1-x^2)x^2}{1-\beta^2x^2} = \frac{(1-\beta^{-2})x^2}{1-\beta^2x^2} + \beta^{-2}x^2 = \frac{(1-\beta^{-2})\beta^{-2}}{1-\beta^2x^2} - \beta^{-2}(1-\beta^{-2}) + \beta^{-2}x^2$$

$$\int_{-1}^{1}\frac{dx}{1-\beta^2x^2} = \frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} \to \int_{\dots} = \frac{1-\beta^{-2}}{\beta^3}\ln\frac{1+\beta}{1-\beta} + \frac{2}{3\beta^2} - \frac{2(1-\beta^{-2})}{\beta^2}$$

$$= \frac{\beta^2-1}{\beta^5}\ln\frac{1+\beta}{1-\beta} + \frac{4}{\beta^4} - \frac{4}{3\beta^2}$$

$$\frac{d\mathcal{E}}{d\omega} = \frac{Z^2}{8\pi c}\left[4 - \frac{4}{3}\beta^2 + \frac{\beta^2-1}{\beta}\ln\frac{1+\beta}{1-\beta}\right] \qquad \beta = v/c$$

Problem 4 relativistic Larmor formula $P = \frac{2e^4}{3c^3}\,\frac{\left(E+\frac{v}{c}\times H\right)^2 - \frac{1}{c}(v\cdot E)^2}{m^2(1-v^2/c^2)}$

$$P = \frac{2e^2}{3c^3}\,\frac{e^2E^2}{m^2(1-v^2/c^2)}$$ ← $H=0,\ v\perp E$

$E_{rad} = Pt$, t = flight time in the capacitor

$P_{rad} = \frac{\vec{v}}{c^2}E_{rad}$

Problem 5

a) $\frac{dp}{dt} = \frac{e}{c} v \times H \qquad \frac{dE}{dt} = \frac{2e^2}{3m^2c^3} \frac{\left(\frac{1}{c} v \times H\right)^2}{1 - v^2/c^2}$

relate v with kinetic energy

$\left(\frac{\mathcal{E}_k}{mc^2}\right)^2 = \frac{1}{1 - v^2/c} \rightarrow \frac{v^2}{c^2} = 1 - \left(\frac{mc^2}{\mathcal{E}_k}\right)^2$

$$\frac{dE}{dt} = \frac{2e^2}{3m^2c^3} H^2 \left(\left(\frac{\mathcal{E}_k}{mc^2}\right)^2 - 1\right)$$

b) $P_r = \frac{dE_{rad}}{dt} = \frac{2e^2}{3m^2c^3} \frac{E_0^2}{1 - v^2/c^2}, \quad S = \frac{c}{4\pi} E_0^2$

$$\sigma = \frac{P_r}{S} = \frac{8\pi e^2}{3m^2c^4} \frac{1}{1 - v^2/c^2}$$

find v for relativistic circular motion

$\omega p = eE_0 \qquad \mathcal{E}_K = (p^2c^2 + m^2c^4)^{1/2} = \left(\left(\frac{eE_0}{\omega}\right)^2 + m^2c^2\right)^{1/2} c$

$$\frac{1}{1 - v^2/c^2} = 1 + \frac{e^2E_0^2}{\omega^2m^2c^4}$$

$$\sigma = \frac{8\pi e^2}{3m^2c^4}\left(1 + \frac{e^2E_0^2}{\omega^2m^2c^4}\right)$$

Correct answers:

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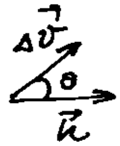
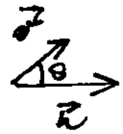
(1) (a) $$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{\omega^2}{\omega_0^2-\omega^2}\right)^2$$

(b) $$\sigma = \frac{8\pi}{3}\left(\frac{\omega^2 a^3}{c^2}\right)^2\left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2,$$ a - radius

(2) (a) $$\frac{d^2E}{d\Omega\, d\omega} = \frac{e^2}{4\pi^2 c^3}\left(\vec{v}_2-\vec{v}_1\right)^2 \sin^2\theta$$



(b) $$\frac{d^2E}{d\Omega\, d\omega} = \frac{e^2}{\pi^2 c^3}\vec{v}^2 \sin^2\theta$$



(3) (a) $$\frac{d^2E}{d\Omega\, d\omega} = \frac{z^2e^2v^4}{4\pi^2c^5}\;\frac{\sin^2\theta\cos^2\theta}{\left[1-\left(\frac{v}{c}\cos\theta\right)^2\right]^2}$$

(b) $$\frac{dE}{d\omega} = \frac{z^2e^2}{2\pi c}\left(-3+\frac{3-(v/c)^2}{v/c}\,\mathrm{Arctanh}\,\frac{v}{c}\right)$$

$$\mathrm{Arctanh}\,\frac{v}{c} = \frac{1}{2}\ln\frac{c+v}{c-v}$$

(4)

(a) $$\delta E'_{rad} = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{eE_0\gamma}{m}\right)^2 dt' = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{eE_0}{m}\right)^2\gamma\, dt$$

$\delta P'_{rad} = 0$ $\delta E', \delta P'$ in comoving frame.

(b) $$\delta E_{rad} = \gamma\left(\delta E'_{rad} + \vec{v}_0\,\delta\vec{P}'_{rad}\right) = \gamma\,\delta E'_{rad} = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{eE_0\gamma}{m}\right)^2 dt$$

$$\delta\vec{P}_{rad} = \gamma\left(\delta P'_{rad} + \frac{\vec{v}_0}{c^2}\,\delta E'_{rad}\right) = \gamma\frac{\vec{v}_0}{c^2}\,\delta E'_{rad} = \frac{\vec{v}_0}{c^2}\cdot\frac{2}{3}\frac{e^2}{c^3}\left(\frac{eE_0\gamma}{m}\right)^2 dt$$

(5) (a) $$\frac{dE}{dt} = \frac{2}{3}\left(\frac{e^2}{mc^2}\right)^2\frac{H^2}{m^2c^3}\left(E^2-m^2c^4\right), \quad \omega = \frac{ecH}{E}$$

(b) $$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\cdot\left(1+\left(\frac{eE_0}{m\omega_0 c^2}\right)^2\right)$$