Problem 1

Start with momentum conservation

$$\frac{\partial}{\partial t} G_k + \nabla_j T_{kj} + f_k = 0$$

$\times\ \varepsilon_{ilk} x_l$

$$\frac{\partial}{\partial t} \underbrace{(r\times G)_i}_{\mathcal{J}_i} + \nabla_j \left(\varepsilon_{ilk} x_l T_{kj}\right) + (r\times f)_i = 0$$

rewrite $\varepsilon_{ilk} x_l T_{kj} = -T_{jk}\varepsilon_{ikl} x_l \equiv$ "$-T\times r$" $= K$

(using symmetry of T, and antisymmetry of ε_{ikl})

Obtain $\frac{\partial}{\partial t}\mathcal{J} + \nabla\cdot K + r\times f = 0$

Problem 2

a) Define $F = E + iB$

$$\nabla\times B = \frac{1}{c}\frac{\partial}{\partial t} E + \frac{4\pi}{c} j_e, \quad \nabla\cdot E = 4\pi\rho_e$$

$$+\ i\times\left\{ -\nabla\times E = \frac{1}{c}\frac{\partial}{\partial t} B + \frac{4\pi}{c} j_m, \quad \nabla\cdot B = 4\pi\rho_m \right.$$

$\Rightarrow$ (A) $-i\nabla\times F = \frac{1}{c}\frac{\partial}{\partial t} F + \frac{4\pi}{c} j$

(B) $\nabla\cdot F = 4\pi\rho$

where $\rho = \rho_e + i\rho_m$, $j = j_e + i j_m$

Eqs (A), (B) are invariant w.r.p.t $\begin{pmatrix} F \\ \rho \\ j \end{pmatrix} \to e^{-i\phi} \times \begin{pmatrix} F \\ \rho \\ j \end{pmatrix}$

Euclidean rotation of components, $e^{-i\phi}\rho = \cos\phi\,\rho_e + \sin\phi\,\rho_m + i(\cos\phi\,\rho_m - \sin\phi\,\rho_e)$

At $\phi = \pi/2$ $\begin{pmatrix} \rho_e \to \rho_m \\ \rho_m \to -\rho_e \end{pmatrix}$; a 90° rotation (the same for F, j, E, B, j_e, j_m)

b) if the ratio $g_K/e_K = \lambda$ is universal, magnetic charge can be eliminated entirely by choosing the rotation angle ϕ so that $\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{g}{e} = \lambda$

Then $g'_K = 0$, $e'_K = (g_K^2 + e_K^2)^{1/2} = (\lambda^2+1)^{1/2} e_K$

Problem 3

$F = E + iB, \quad F^* = E - iB$

(i) $\frac{1}{8\pi} F \cdot F^* = \frac{1}{8\pi}(E + iB)\cdot(E - iB) = \frac{1}{8\pi}(E^2 + B^2) \equiv U$ energy density

(ii) $\frac{1}{8\pi} F \times F^* = \frac{1}{8\pi}(E + iB)\times(E - iB) = \frac{i}{4\pi} E \times B = ic\, G$ ← momentum density

(iii) $\frac{1}{8\pi}(F_i F_j^* + F_i^* F_j) = \frac{1}{4\pi}(E_i E_j + B_i B_j) = \delta_{ij} \cdot U - T_{ij}$ ← stress tensor

Under $F \to e^{-i\phi} F$, all three quantities are <u>invariant</u>.

Problem 4

a) In a uniform field B, the force on a loop

$$\vec{F} = \frac{I}{c}\oint d\vec{\ell} \times \vec{B} = \frac{I}{c}\left(\oint d\vec{\ell}\right)\times \vec{B} = \frac{I}{c}\, 0 \times \vec{B} = 0$$

$$\vec{\tau} = \frac{I}{c}\oint \vec{r}\times(d\vec{\ell}\times\vec{B}) = \frac{I}{c}\oint (r\cdot B)\, d\vec{\ell} - (r\cdot d\ell)\,\vec{B}$$

$$\tau_i = \frac{I}{c}\oint\left(x_j B_j\, d\ell_i - x_k\, d\ell_k\, B_i\right) = \frac{I}{c}\oint B_j\left(x_j\, d\ell_i - x_i\, d\ell_j\right)$$

Have used $\oint x_k\, d\ell_k = 0$, $\oint(x_i\, d\ell_j + x_j\, d\ell_i) = 0$ (★)
(see Chap. 28, Eqs (28.6) (28.7))

However, since $x_j\, d\ell_i - x_i\, d\ell_j = \frac{1}{2}\varepsilon_{jin}(r\times d\ell)_n$, have $\underline{\vec{\tau} = \vec{m}\times\vec{B}}$

b) Choosing r_0 as the origin,

$$F_i = \frac{I}{c}\oint \varepsilon_{ijk}\, d\ell_j\, x_p \nabla_p B_k = \frac{I}{2c}\oint \varepsilon_{ijk}(d\ell_j\, x_p + d\ell_p\, x_j)\nabla_p B_k =$$

$$= -\underbrace{\varepsilon_{ijk}\,\varepsilon_{jpn}}_{\delta_{kp}\delta_{in} - \delta_{kn}\delta_{ip}} m_n \nabla_p B_k = -m_i \underbrace{\nabla\cdot B}_{0} + m_k \nabla_i B_k = \underline{\nabla_i(m B)}$$

Note: $m_n = \frac{I}{2c}\oint(r\times d\ell)_n = \frac{I}{2c}\oint \varepsilon_{npq}\, x_p\, d\ell_q$

Thus $\frac{I}{2c}\oint(d\ell_j\, x_p - d\ell_p\, x_j) = \varepsilon_{pjn}\, m_n$

Problem 5

a) Find $A(r) = \frac{I}{c}\oint \frac{d\vec{\ell}}{|r-r'|}$ for current loop

use Taylor expansion $\frac{1}{|r-r'|} = \frac{1}{r} + \frac{r\cdot r'}{r^3} + O(r^{-3})$

$$A_i(r) = \frac{I}{c}\oint\left(\frac{1}{r} + \frac{x_j x_j'}{r^3}\right) d\ell_i = \frac{I}{2c}\oint x_j(x_j'\, d\ell_i - x_i'\, d\ell_j)\frac{1}{r^3}$$

$$A_i(r) = \frac{x_j}{r^3}\varepsilon_{jik} m_k = \frac{(m\times r)_i}{r^3}$$

Find $B_i = (\nabla\times A)_i = \varepsilon_{ijk}\nabla_j \varepsilon_{kpq}\frac{m_p x_q}{r^3}$

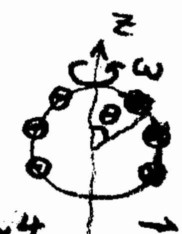
$$B_i = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\left(\frac{m_p\delta_{jq}}{r^3} - 3\frac{m_p x_q x_j}{r^5}\right)$$

$$B_i = \cancel{\frac{m_i \times 3}{r^3} - 3\frac{m_i r^2}{r^5}} - \frac{\delta_{ij} m_j}{r^3} + 3\frac{m_j x_j x_i}{r^5} = \frac{3\hat r_i(m\cdot r) - m_i}{r^3}$$

b) From Prob 4b, $U = -\vec m\cdot\vec B$ for a dipole, thus

$$U_{12} = -\vec m_2 \vec B_2 = \frac{\vec m_1\cdot\vec m_2 - 3(\vec m_1\cdot\hat r_{12})(\vec m_2\cdot\hat r_{12})}{r_{12}^3}$$

Problem 6



a) $\vec j = \rho\vec v = \omega a\sigma\delta(r-a)\sin\theta\,\hat\varphi$

$$m = \frac{1}{2c}\int r\times j = \frac{\omega a^4}{2c}\sigma\int_0^\pi 2\pi\sin^2\theta\, d\cos\theta = \frac{\omega a^2 q}{8\pi c}2\pi\times\frac{4}{3}$$

$$\vec m = \frac{\omega a^2 q}{3c}\hat z$$

b) $\nabla\cdot B = \nabla\times B = 0$ (inside & outside)

$B_\perp$ continuous, $\delta B_\parallel = \frac{4\pi}{c}K$ Try $\vec B = \begin{cases} B_0\hat z & (r<a)\\ \frac{3(m\hat r)\hat r - \vec m}{r^3} & (r>a)\end{cases}$

$B_0 = 2m/a^3$

$\frac{m}{a^3} + B_0 = \frac{4\pi}{c}K \to m = \frac{4\pi K}{3c}a^3 = \frac{\omega a^2 q}{3c}$ agrees

c) $\vec B(r>a) = \frac{3(m\hat r)\hat r - \vec m}{r^3}$ $(j = K\sin\theta\,\delta(r-a))$

d) $\vec G(r>a) = \frac{1}{4\pi c}E\times B = \frac{\omega a^2 q^2}{12\pi c^2}\frac{\sin\theta}{r^5}\hat\varphi$, $\vec G(r<a)=0$

$\vec J = \int(r\times G) = L\hat z$, with $L = \int_{r>a} 2\pi r^2 dr \frac{\omega a^2 q^2}{12\pi c^2}\frac{\sin^2\theta}{r^4}|d\cos\theta| = \frac{\omega a q^2}{6c^2}\times\frac{4}{3} = \frac{2}{9}\frac{\omega a q^2}{c^2}$