# Problem Set #5 8.311

**Prob 1** Force for static fields

a) $F_i = \rho E_i + \frac{1}{c}(j\times B)_i = \frac{1}{4\pi}(\nabla\cdot D)E_i + \frac{1}{4\pi}\left[\left(\nabla\times H - \cancel{\frac{1}{c}\frac{\partial D}{\partial t}}\right)\times B\right]_i$

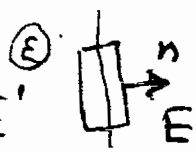
add $0 = \frac{1}{4\pi}(\nabla\cdot B)H_i + \frac{1}{4\pi}\left[(\nabla\times E)\times D\right]_i$

$F_i = \frac{1}{4\pi}(\nabla\cdot D)E_i + \frac{1}{4\pi}\left((\nabla\times E)\times D\right)_i + \left[E, D\to H, B\right]$

$D = \varepsilon E$, $B = \mu H$

$F_i = -\nabla_i \frac{\varepsilon E^2}{8\pi} + \frac{1}{4\pi}\nabla_j(\varepsilon E_i E_j) - \nabla_i \frac{\mu H^2}{8\pi} + \frac{1}{4\pi}\nabla_j(\mu H_i H_j) \equiv -\nabla_j T_{ji}$

$F_i^{tot} = \int F_i(dr) = -\oint T_{ij}\, da_j$

b) 


$\vec F = -\frac{E^2 - \varepsilon(E')^2}{8\pi}\vec n + \frac{\vec E}{4\pi}(E\cdot n) - \frac{\varepsilon \vec E'}{4\pi}(\vec E'\cdot\vec n)$

(i) $E \parallel n$, $E' = \frac{1}{\varepsilon}E$, $\vec F = \frac{(1-1/\varepsilon)}{8\pi}E^2\hat n$

(ii) $E \perp n$, $E' = E$, $F = \frac{\varepsilon - 1}{8\pi}E^2 n$

Elementary appoach: use the method of virtual work

$\delta W = \int \delta\left(\frac{DE}{8\pi} + \rho\phi\right) d^3r \underset{D=\varepsilon E}{=} \ldots = \int \delta r\left\{\frac{1}{8\pi}E^2\nabla\varepsilon - \frac{1}{8\pi}\nabla\left(E^2\frac{\partial\varepsilon}{\partial\tau}\tau\right) - \rho E\right\} d^3r$ (τ: density)

$\vec f = -\frac{1}{8\pi}E^2\nabla\varepsilon + \rho E + \frac{1}{8\pi}\nabla\left(E^2\frac{\partial\varepsilon}{\partial\tau}\tau\right)$

Ignoring the last term, $f_i = -\frac{1}{8\pi}E^2\nabla_i\varepsilon + \frac{1}{4\pi}E_i\nabla D$

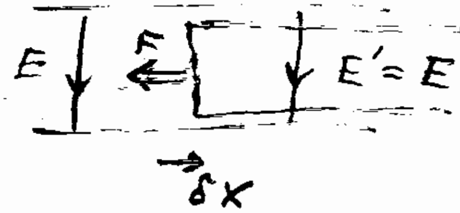
$f_i = \nabla_j \frac{E_i D_j}{4\pi} - \frac{D_j}{4\pi}\nabla_j E_i - \frac{E^2}{8\pi}\nabla_i\varepsilon$ Note $(D\cdot\nabla)E_i = \frac{\varepsilon}{2}\nabla_i E^2$ since $\nabla\times E = 0$

Obtain $f_i = \nabla_j\left(\frac{D_j E_i}{4\pi}\right) - \nabla_i\frac{DE}{8\pi} = -\nabla_j T_{ij}$

where $T_{ij} = \frac{DE}{8\pi}\delta_{ij} - \frac{D_i E_j}{4\pi}$

Both methods of calculating force yield inentical results.

Prob 2 a)




$$E = \frac{V}{d}$$

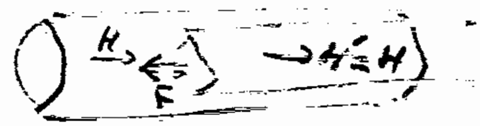
$$F_{tot} = \frac{\varepsilon - 1}{8\pi} E^2 A = \frac{\varepsilon-1}{8\pi}\left(\frac{V}{d}\right)^2 L d$$

Alternative soln. by method of virtual work

$$F = -\frac{\delta U}{\delta x} + \delta Q V = -\frac{1}{2} V \frac{dC}{dx}, \quad C = \frac{L^2}{4\pi d}\left(\varepsilon\left(1 - \frac{x}{L}\right) + \frac{x}{L}\right)$$

$$F = -\frac{\varepsilon - 1}{8\pi} \frac{V^2 L}{d}$$

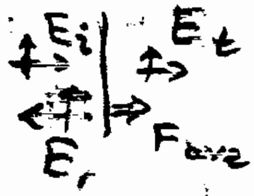
b)




$\nabla \times H = \frac{4\pi}{c} j \rightarrow H = \frac{4\pi}{c}\frac{NI}{L}$ everywhere in the solenoid

$$F = -\pi r^2 \frac{B'H' - BH}{8\pi} = -\frac{\mu - 1}{8} r^2 H^2 = -2\pi^2 \frac{\mu - 1}{c^2} \frac{I^2 N^2}{L^2}$$

Prob 3 a)




$$F = \frac{(E_i + E_r)^2}{8\pi} - \frac{E_t^2}{8\pi} + \frac{(B_i + B_r)^2}{8\pi} - \frac{B_t^2}{8\pi}$$

$$B_r = -E_r, \quad B_t = E_t, \quad B_i = E_i$$

$$F = \frac{1}{4\pi}\left(E_i^2 + E_r^2 - E_t^2\right) = \frac{2R}{4\pi} E_0^2 \cos^2(\omega t)$$

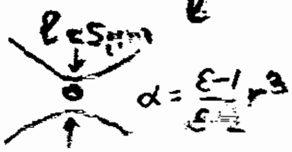
time-ave: $\times \frac{1}{2}$ $\quad F_{ave} = \frac{R}{4\pi} E_0^2$

b) $$F = \frac{(E_i + E_r)^2}{8\pi} - \varepsilon \frac{E_t^2}{8\pi} + \frac{(B_i + B_r)^2}{8\pi} - \frac{B_t^2}{8\pi} = \frac{E_i^2 + E_r^2 - \varepsilon E_t^2}{4\pi}$$

$B_t = \sqrt{\varepsilon} E_t$ $$F = \frac{E_i^2}{4\pi}\left(1 + \left(\frac{1-n}{1+n}\right)^2 - \frac{(2n^2)^2}{(1+n)^2}\right) = \frac{2E_i^2}{4\pi}\frac{(1-n^2)(1+2n^2)}{(1+n)^2}$$

time-ave $2E_i^2 \rightarrow E_0^2$ $$F_{ave} = \frac{E_0^2}{4\pi}\frac{(1-\varepsilon)(1+2\varepsilon)}{1+\varepsilon}$$

c) $F \sim \frac{U}{\ell} = \frac{\frac{1}{2}\alpha \overline{E^2}}{\ell}$; Power $W = \frac{\pi \ell^2}{4} \bar{c} \cdot \frac{\varepsilon^2}{4\pi} c \overline{E^2}$ $\quad W \sim mgc \sim 1\,\text{mWatt}$.




$\alpha = \frac{\varepsilon - 1}{\varepsilon + 2} r^3$ $\quad F \sim \frac{\pi}{2 \cdot} \frac{16 W}{c \ell^2} \sim 8 \frac{W}{c} = mg$ (balance gravity)

**Prob 4** a) $S_r = \left|\frac{1-n}{1+n}\right|^2 S_i$, $S_{tot} = \left(1 - \left|\frac{1-n}{1+n}\right|^2\right) S_i \underset{\uparrow}{=} 4\sqrt{\frac{\omega}{8\pi\sigma}}\, S_i$

$n = \sqrt{1 + \frac{4\pi\sigma}{\omega} i} \underset{\underset{\omega \ll \sigma}{\uparrow}}{=} \left(\frac{4\pi\sigma}{\omega} i\right)^{\frac{1}{2}}$

b) $\underset{x>0}{E(t)} = \frac{2}{n+1} \cdot e^{ikx - i\omega t}$ $\quad k = \frac{\omega}{c} n$

$B = nE \qquad S = \frac{c}{4\pi} (E \times B)_x$

**Prob 5** $\quad G(\tau) = \int_{-\infty}^{\infty} \frac{\omega_p^2 e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}\frac{d\omega}{2\pi} = \begin{cases} \oint \ldots = 0 & \tau<0 \\ \oint\ldots & \tau>0 \end{cases}$

$$G(\tau>0) = \frac{-1}{2\pi}\oint \frac{\omega_p^2 e^{-i\omega\tau}\,d\omega}{(\omega-\omega_+)(\omega-\omega_-)} = \frac{2\pi i}{2\pi}\left[\frac{\omega_p^2 e^{-i\omega_+\tau}}{\omega_+-\omega_-} + \frac{\omega_p^2 e^{-i\omega_-\tau}}{\omega_--\omega_+}\right]$$

$$G(\tau>0) = \frac{i\omega_p^2}{2\omega_1}\left(e^{-i\omega_+\tau} - e^{-i\omega_-\tau}\right)$$

$$G(\tau>0) = \frac{\omega_p^2}{\omega_1} e^{-\gamma\tau/2}\sin\omega_1\tau \qquad \omega_\pm = -\frac{i\gamma}{2}\pm\omega_1,\ \omega_1^2=\omega_0^2-\gamma^2/4$$

a) Expand $G(\omega)$ in powers of $\omega$:

$$G(\omega) = 1-\frac{\omega_p^2}{(\omega-\omega_-)(\omega-\omega_+)} = 1-\frac{\omega_p^2}{\omega_+-\omega_-}\left[\frac{1}{\omega_+-\omega} - \frac{1}{\omega_--\omega}\right]$$

$$G(\omega) = 1-\frac{\omega_p^2}{2\omega_1}\sum_{n\geq 0}\left(\frac{\omega^n}{\omega_+^{n+1}} - \frac{\omega^n}{\omega_-^{n+1}}\right)$$

In time domain, $\omega \to i\partial_t$, obtain

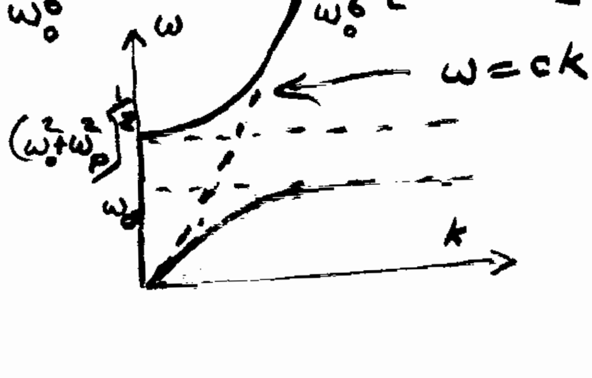
$$D(t) = E(t) + \sum_{n\geq 0} g_n \partial^n E/\partial t^n$$

$$g_0 = -\frac{\omega_p^2}{2\omega_1}\left(\omega_+^{-1}-\omega_-^{-1}\right) = \frac{\omega_p^2}{\omega_0^2}$$

$$g_1 = -i\frac{\omega_p^2}{2\omega_1}\left(\omega_+^{-2}-\omega_-^{-2}\right) = i\omega_p^2\frac{-i\gamma}{\omega_0^4} = \gamma\frac{\omega_p^2}{\omega_0^4}$$

$$g_2 = \frac{\omega_p^2}{2\omega_1}\left(\omega_+^{-3}-\omega_-^{-3}\right) = \omega_p^2\frac{\omega_+^2+\omega_-^2+\omega_+\omega_-}{\omega_0^6} = \frac{\omega_p^2}{\omega_0^6}\left[-\gamma^2+\omega_0^2\right]$$

b) $$k^2 = \frac{\omega^2}{c^2}\left(1+\frac{\omega_p^2}{\omega_0^2-\omega^2}\right)$$




$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{\omega}{k}\frac{dk^2}{d\omega^2}$$

$$v_g^{-1} = \frac{\omega}{kc^2}\left[1+\frac{\omega_p^2\omega^2}{(\omega_0^2-\omega^2)^2}\right]$$

Prob 6 a) KK for Re $\varepsilon$:

$$\mathrm{Re}\,\varepsilon(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{\omega'\,\mathrm{Im}\,\varepsilon(\omega')\,d\omega'}{\omega'^2-\omega^2} = 1 - \frac{G}{\omega^2} + O\left(\frac{1}{\omega^4}\right)$$

Compare to $\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} + O(\omega^{-4})$

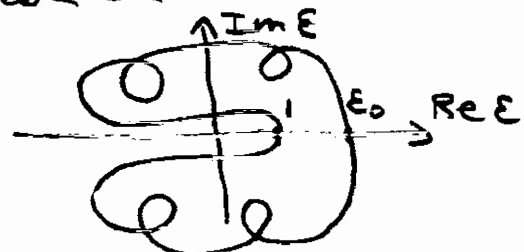
$$\omega_p^2 = \frac{2}{\pi}\int_0^\infty \omega'\,\mathrm{Im}\,\varepsilon(\omega')\,d\omega'$$

b) (i) $\varepsilon(\omega) = \varepsilon(-\omega^*)^*$ follows from Cauchy thm representation $\varepsilon(\omega) = \int_{-\infty}^{\infty} \frac{\varepsilon(\omega')}{\omega-\omega'+i0}\frac{d\omega'}{2\pi}$

for $\omega = i\lambda$, it yields

$\varepsilon(i\lambda) = \varepsilon(-i\lambda)^* \rightarrow \varepsilon(i\lambda)$ is real

(ii) Map the path $-\infty<\omega<\infty$ in $\omega$-plane onto $\varepsilon$ plane:




The only three points with $\mathrm{Im}\,\varepsilon=0$ are:

$\omega = \pm\infty \rightarrow \varepsilon = 1$

$\omega = 0 \rightarrow \varepsilon = \varepsilon_0 > 1$

symmetry follows from $\varepsilon(\omega^*) = \varepsilon(\omega)^*$

By argument principle, $\varepsilon(\omega)$ takes any value $1<\omega<\varepsilon_0$ and no other real values

Combined with (i), this indicates that $\varepsilon(\omega)$ is monotonic on the imaginary axis

c) see b)

d) KK for $\frac{1}{\varepsilon}$ arises the same way as for $\varepsilon$

$$\mathrm{Re}\,\frac{1}{\varepsilon} = 1 + \frac{2}{\pi}P\int_0^\infty \frac{\omega'\,\mathrm{Im}\,\varepsilon^{-1}(\omega')}{\omega'^2-\omega^2}\,d\omega'$$

$$\mathrm{Im}\,\frac{1}{\varepsilon} = -\frac{2\omega}{\pi}P\int_0^\infty \frac{\mathrm{Re}\,\varepsilon^{-1}(\omega')\,d\omega'}{\omega'^2-\omega^2}$$

sum rule

$$\omega_p^2 = -\frac{2}{\pi}\int_0^\infty \omega'\,\mathrm{Im}\,\frac{1}{\varepsilon(\omega')}\,d\omega'$$