3. Angular momentum

3.1 SO(3) vs. SU(2)

We are interested in studying <u>rotational symmetry group</u> & its representations.

[group G: closed $gh \in G$, unit $1 \cdot g = g$, inverses $g^{-1}g = gg^{-1} = 1$, associative $f(gh) = f(g)h$]

What is rotational symmetry group?

Natural candidate: SO(3), rotation group of $\mathbb{R}^3$

SO(3): 3×3 (special) orthogonal matrices

$$R^T R = \mathbb{1} \qquad \text{(preserve inner product } \vec{a}\cdot\vec{b}\text{)}$$
$$\det R = 1 \qquad \text{(preserve orientation)}$$

Examples:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Can get any rotation in SO(3) by multiplying these.

Note: $R_x(\alpha) R_z(\beta) \neq R_z(\beta) R_x(\alpha)$

<u>nonabelian group</u>

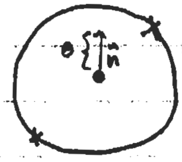
Any rotation can be characterized by:

$\hat{n}$ axis of rotation
θ angle

$SO(3)$ is a 3-dimensional manifold (looks like $\mathbb{R}^3$ locally)
(circle bundle over $\mathbb{RP}^2$)

A group which is a manifold is called a <u>Lie group</u>

Picture:



Ball in $\mathbb{R}^3$ of radius π, identify $(\hat{n}, \pi) \sim (-\hat{n}, \pi)$.

Seems like this is rotational symmetry group.

BUT...

Consider neutron interferometer (PS 9, prob 4.)

In B field $\vec{B} = B\hat{z}$,
neutron with magnetic moment $\frac{ge\hbar}{2mc}$ has coupling
$(g \sim -1.91)$

$$H = \frac{ge}{2mc}\vec{S}\cdot\vec{B}$$

$$= \omega S_z, \qquad \omega = \frac{geB}{2mc}$$

So if at $t = 0$, state is $\chi(0) = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$ $[c_+|+\rangle + c_-|-\rangle]$

At time t, state is

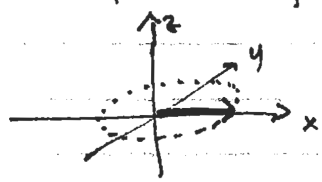
$$\chi(t) = e^{-\frac{iH}{\hbar}t}\chi(0) = \begin{pmatrix} e^{-\frac{i\omega t}{2}} c_+ \\ e^{\frac{i\omega t}{2}} c_- \end{pmatrix}.$$

Describes <u>precession</u> of spin, with angular frequency ω.

Ex. start in state with $S_x = +\hbar/2$

$$\chi(0) = |S_x, +\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

At time t, $\chi(t) = |S_n, +\rangle$, $\hat{n} = \hat{x}\cos\omega t + \hat{y}\sin\omega t$
up to a phase.



After time $T_- = 2\pi/\omega$,

$$\chi(T_-) = \begin{pmatrix} -c_+ \\ -c_- \end{pmatrix} = -\chi(0) = -|S_x, +\rangle.$$

State has rotated once, again has $S_x = +\hbar/2$.

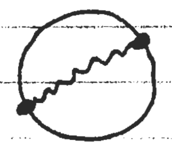
<u>But</u> appearance of phase (-1) changes interference pattern!

To get successive maxima at same point,

need $T_+ = \dfrac{4\pi}{\omega}$ $\qquad\left[\Delta B = \dfrac{4\pi\hbar c}{|e| g \chi \ell}\right]$

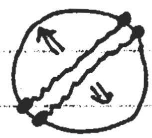
This demonstrates that rotation by $360°$ is not always a trivial transformation.

In SO(3), rotation by 2π cannot be deformed into a trivial transformation.



[Demo]

But rotation by 4π can be.



[Demo]

[Technically, $\pi_1(SO(3)) = \mathbb{Z}_2$]

This leads us to consider a "larger" group: SU(2).

SU(2): 2×2 (special) unitary matrices

$$U^\dagger U = \mathbb{1} \qquad \text{(preserve inner product } \chi^\dagger \chi)$$
$$\det U = 1.$$

General SU(2) matrix:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad a, b \in \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1.$$

SU(2) is group describing rotations of an electron (spin ½) state.

Topologically, $SU(2) \cong S^3$, since $|a|^2 + |b|^2 = 1$ describes a sphere in $\mathbb{R}^4 = \mathbb{C}^2$.

All loops in S^3 are contractible, unlike in SO(3).

Can map $SU(2) \longrightarrow SO(3)$ by group homomorphism
$\pm 1 \longrightarrow 1$.

write $SO(3) = SU(2)/\mathbb{Z}_2$

For example,

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$SU(2)$ is simply connected, "universal covering group" of $SO(3)$.

3.2. Lie algebra & representations of $SU(2)$.

We want to understand how symmetry group $SU(2)$ works in QM.

Symmetry group acts through representations on $\mathcal{H}$.

[Representation $\mathcal{D}$:

$\mathcal{D}(g) : \mathcal{H} \to \mathcal{H}$ linear map $\forall g$

$\mathcal{D}(1) = \mathbb{1}$.

$\mathcal{D}(gh) = \mathcal{D}(g)\mathcal{D}(h)$]

To understand representations of a Lie group, consider <u>Lie Algebra</u>

Associated with a Lie group G is an Lie algebra $\underline{\mathfrak{g}}$, of infinitesimal elements of G.

For example, for SO(3)

$\mathbb{1} + \varepsilon A$ is orthogonal if (working to order ε)

$$(\mathbb{1} + \varepsilon A)(\mathbb{1} + \varepsilon A^T) = \mathbb{1} + \varepsilon(A + A^T) = \mathbb{1},$$

so $A = -A^T$.

Basis of Lie algebra for SO(3) given by

$$K_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$K_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$K_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For QM, want Hermitian operators, so write

$$J_\ell = i\hbar K_\ell.$$ [Note: will change basis later so J_z is diagonal.]

Lie algebra defined by on space spanned by J_i $[A, B]$; J_i are generators of algebra.

Properties: of a general Lie algebra
i) Closed $[J_i, J_j] = i f_{ijk} \hbar J_k$ — structure constants
ii) linear in A, B
iii) $[A, B] = -[B, A]$
iv) $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$ (Jacobi)

Same algebra for SO(3), SU(2): $f_{ijk} = \varepsilon_{ijk}$

$$[J_i, J_j] = i\varepsilon_{ijk}\hbar J_k$$
$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k. \qquad \left(S_i = \frac{\hbar}{2}\sigma_i\right)$$

Any element of $\begin{matrix} SU(2) \\ SO(3) \end{matrix}$ can be written as

$$g = e^{-\frac{i}{\hbar}\left(\begin{smallmatrix}\vec{\sigma}\cdot \\ \vec{J}\cdot\end{smallmatrix}\hat{n}\right)\phi}$$

for g = rotation by ϕ about $\hat{n}$.

when $\phi = 2\pi$, $g = \mathbb{1}$ in $SO(3)$,
$g = -\mathbb{1}$ in $SU(2)$.

Representations of a Lie algebra:

$$\mathcal{D}(k) : \mathcal{H} \to \mathcal{H} \qquad \forall k \in \mathfrak{g}, \mathcal{D} \text{ linear in } k.$$
$$\mathcal{D}(0) = 0$$
$$\mathcal{D}([k,l]) = [\mathcal{D}(k), \mathcal{D}(l)]$$

To each representation of the group, there is a corresponding representation of the algebra (but not necessarily vice-versa if gp not simply connected.)

Classify representations of group by representations of the algebra.

[Notation: write $J_i = \mathcal{D}[\sigma_i]$ on general rep. space $\mathcal{H}$]

Representations of $SU(2)$ algebra: $\quad [J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$.

Define
$$J^2 = J_x^2 + J_y^2 + J_z^2$$
$$J_\pm = J_x \pm i J_y.$$

Can show:
$$[J^2, J_i] = 0$$
$$[J_z, J_\pm] = \pm\hbar J_\pm$$
$$[J_+, J_-] = 2\hbar J_z$$

and $J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = J_z^2 + J_-J_+ + \hbar J_z$

with $J_\pm^+ = J_\mp$

Can simultaneously diagonalize J^2, J_z.

write
$$J^2|a,b\rangle = a|a,b\rangle$$
$$J_z|a,b\rangle = b|a,b\rangle$$

What values of a, b are allowed?

$$\langle a,b|\underbrace{J^2}_{a}|a,b\rangle = \langle a,b|\underbrace{J_z^2}_{b^2} + \tfrac{1}{2}(\underbrace{J_+J_-}_{\geq 0} + \underbrace{J_-J_+}_{\geq 0})|a,b\rangle$$

$$\Rightarrow \quad a \geq b^2$$

compare
$$[J_z, J_\pm] = \pm\hbar J_\pm,$$
$$[N, {}^{a^+}_{a}] = \pm \hbar\, {}^{a^+}_{a}.$$

so $J_\pm$ are raising/lowering operators for J_z.

$$J_z(J_\pm|a,b\rangle) = (b\pm\hbar)(J_+|a,b\rangle).$$

but
$$[J^2, J_\pm] = 0, \quad \text{so}$$
$$J_\pm|a,b\rangle = C_\pm^{(a,b)}|a, b\pm\hbar\rangle$$

Since $a \geq b^2$, there must be a maximum b which can be reached for a fixed a. Call this $b_{max} = \hbar j$.

Then
$$\langle a,b|J_-J_+|a,b\rangle = \langle a,b|J^2 - J_z^2 - \hbar J_z|a,b\rangle$$
$$|C_+(a,b)|^2 = a - b^2 - \hbar b.$$

This must vanish for $b_{max} = \hbar j$. so $a = \hbar^2 j(j+1)$.

Angular momentum: So far

- Considered groups $SO(3)$ & $SU(2)$.
  In $SO(3)$, rotation by 2π $\rightarrow$ 1
  " $SU(2)$ " " " $\rightarrow$ -1
  in physics, sometimes " " " $\rightarrow$ -1 (neutron interferometer)

Conclude: $SU(2)$ is more fundamental for physics
(can map $\pm 1 \rightarrow 1$, not other way around).

- Want to understand representations $\mathcal{D}(g): \mathcal{H} \rightarrow \mathcal{H}$ of $SU(2)$.
  Simplest to work with algebra.

$$1 + i\varepsilon A \quad \text{in } SU(2) \text{ if}$$
$$(1 + i\varepsilon A)(1 - i\varepsilon A^{+}) = 1 + \mathcal{O}(\varepsilon^2)$$
$$\Rightarrow \quad A = A^{+}.$$

Basis for 2×2 Hermitian matrices:
$$S_i = \hbar/2 \, \sigma_i$$

Near 1, structure of group contained in algebra structure

$$[S_i, S_j] = i\hbar\varepsilon_{ijk} S_k.$$

Want to find representation of algebra

$$J_i = \mathcal{D}(S_i) \quad : \mathcal{H} \rightarrow \mathcal{H}.$$

so $$[J_i, J_j] = i\hbar \, \varepsilon_{ijk} J_k$$

Last time: constructed all irreducible representations of $SU(2)$

Similarly, must be a b_{min} which can be reached by acting with J_-.

$$|C_-(a,b)|^2 = a - b^2 + \hbar b$$

so $b_{min} = -b_{max}$.

It follows that $2b_{max} = n\hbar$, so $j = n/2$ is half-integral.

For each $n = 2j, \in \mathbb{Z}$ we have constructed an irreducible n-dimensional representation $\mathcal{H}_j$ of su(2) algebra. $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$

$\mathcal{H}_j$ spanned by $\{|j,m\rangle, \ m = -j, -j+1, \ldots, j-1, j\}$

$$J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \qquad J_z|j,m\rangle = m\hbar|j,m\rangle$$

$$J_+|j,j\rangle = J_-|j,-j\rangle = 0$$

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,|j,m\pm 1\rangle.$$

(Irreducible representation: no linear subspace is closed under the action of all J_i's.)

Can use representations of algebra to get group representation through

$$\mathcal{D}^{(j)}_{m'm}(g) = \langle j,m'|e^{-\frac{i}{\hbar}(\vec{J}\cdot\hat{n})\phi}|j,m\rangle, \qquad g = e^{-\frac{i}{\hbar}(\vec{J}\cdot\hat{n})\phi}$$

$\mathcal{D}^{(j)}_{m'm}(g)$ are Wigner functions on group G.

Theorem: dim. n irrep is unique up to unitary isomorphism.

Today: specific representations, spherical harmonics.

First: comments:

~~[illegible]~~

Lie groups & representation theory:

19th century mathematics (Lie, Cartan, etc)

Became important for particle physics in 50's.

"Isotopic spin" (isospin): (almost) symmetry of p, n in fundamental representation of SU(2).

$j = 1/2$ n (udd)   p (uud)

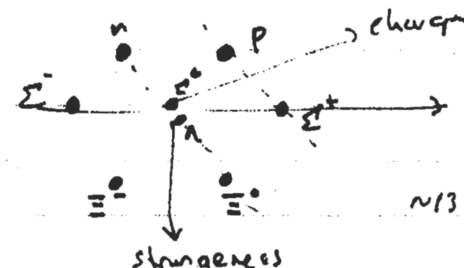
other multiplets

$j = 1$ π^- π^0 π^+

⋮

Suggested in 50's: strange particles included by extending to SU(3) (with strangeness corrections)

Basic multiplet of SU(3): (octet) $J^P = \frac{1}{2}^+$



~939 MeV

~1190, 1110

~1318

axes ~ generalizations of J_z $\begin{pmatrix}1 & \\ & -1 & \\ & & 0\end{pmatrix}, \begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}, \begin{pmatrix}0 & & \\ & 1 & \\ & & -1\end{pmatrix}$

Decuplet: ($J^P = \frac{3}{2}^+$)

Δ^- Δ^0 Δ^+ Δ^{++} ~1200 MeV

Σ^- Σ^0 Σ^+ ~1380 MeV

Ξ^- Ξ^0 ~1530 MeV

Ω^- (1672) → predicted by group theory 3 years before discovery!

Georgi: Lie algebras in particle physics

SU(2):

Specific representations, $j =$ "spin" of representation

$j=0$: only state is $|j,m\rangle = |0,0\rangle$

$$J^2|0,0\rangle = J_\pm|0,0\rangle = J_i|0,0\rangle = 0.$$

action of any group element is trivial $\mathcal{D}^{(0)}(g)\,|0,0\rangle = |0,0\rangle$.

$j = 1/2$ (spin-1/2 system)

States $|j,m\rangle = |1/2, \pm 1/2\rangle$. (previously: $|\pm\rangle$ or $|S_z;\pm\rangle$)

$$J_i = S_i = \frac{\hbar}{2}\sigma_i, \qquad J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{D}^{(1/2)}(\hat{n},\phi) = \exp\left[-i(\vec{n}\cdot\vec{\sigma})\phi/2\right] = \left(\cos\frac{\phi}{2}\right)\mathbb{1} + i\sin\frac{\phi}{2}\,(\vec{n}\cdot\vec{\sigma}),$$

as discussed in earlier lectures.

[This gives background for examples previously described].

$j=1$: (spin 1)

States $|1,1\rangle, |1,0\rangle, |1,-1\rangle$.

$$J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad J_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Note: looks different from $J_i = i\hbar K_i$ above,
since in this basis $\underline{J_z \text{ is diagonal}}$. Otherwise, just
<u>related</u> by orthogonal change of basis.

- For general j, if $j \overset{=l}{\in} \mathbb{Z}$, representation of SO(3) since $e^{-i/\hbar 2\pi J_i} = \mathbb{1}$.
if $j + 1/2 \in \mathbb{Z}$, $e^{i/\hbar 2\pi J_i} = -\mathbb{1}$, not a rep. of SO(3).

3.3 Spherical harmonics

Consider functions on $S^2 = \{x, y, z : x^2 + y^2 + z^2 = 1\}$,
(parameterized by θ, ϕ.)

An SO(3) rotation gives a linear transformation on the set of functions on S^2. The set of homogeneous polynomials of degree N (in x,y,z) is invariant under so(3) & must form a representation of so(3), hence of SU(2). (with $j \in \mathbb{Z}$)

Counting functions:

		total
constant :	1	1
linear :	x, y, z	3
quadratic :	$x^2, y^2, z^2, xy, yz, zx$	$6 = 5 + 1$ from $x^2+y^2+z^2$

Total # of independent polynomials of degree N:

$$\sum_{\substack{K\text{ odd}\\ K=1}}^{2N+1} K = (N+1)^2$$

Theorem:
At degree N, acquire $2N+1$ polynomials living in a spin $j=N$ (irreducible) representation.

Associated eigenfunctions of J^2, J_z: $Y_{lm}(\theta, \phi)$

Can explicitly construct $Y_{lm}(\theta,\phi)$ from general representation theory.

Defining $\vec{L} = \vec{r} \times \vec{p}$, generators of so(3)

$$L^i = -i\hbar \, \varepsilon_{ijk} \, x^j \frac{\partial}{\partial x^k},$$

have

$$L_z = \frac{\hbar}{i} \, \partial/\partial\phi$$
$$L_\pm = \hbar e^{\pm i\phi}\left(i \cot\theta \frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right).$$
$$L^2 = -\hbar^2\left[\csc^2\theta \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right)\right].$$

Looking for functions solving

$$L^2 Y_{lm}(\theta,\phi) = \hbar^2 l(l+1) \, Y_{lm}(\theta,\phi)$$
$$L_z Y_{lm}(\theta,\phi) = \hbar m \, Y_{lm}(\theta,\phi).$$

(prop. to assoc. Legendre polys)

From L_z clearly $Y_{lm}(\theta,\phi) = e^{im\phi} P_{lm}(\theta)$.

$$L_+ Y_{ll}(\theta,\phi) = \hbar e^{i(l+1)\phi}\left[-l\cot\theta + \frac{\partial}{\partial\theta}\right] P_{lm}(\theta) = 0$$

$$\Rightarrow \quad P_{ll} = \text{const.} \, (\sin\theta)^l$$

so $$Y_{ll} = c_l e^{im\phi} (\sin\theta)^l$$

Normalization $$\underbrace{\int d\Omega}_{\int_0^{2\pi} d\phi \int_0^\pi d\theta} \, |Y_{lm}(\theta,\phi)|^2 = 1$$

$$\Rightarrow \quad c_l = \frac{(-1)^l}{2^l \, l!}\sqrt{\frac{(2l+1)\;(2l)!}{4\pi}}$$ [sign by convention]

Generate Y_{lm} by acting with L_-

$$Y_{lm}(\theta,\phi) = \frac{(-1)^l}{2^l \, l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}} \, e^{im\phi} \frac{1}{\sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l}$$

Exs: $Y_{00} = \frac{1}{\sqrt{4\pi}}$ ((constant))

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi} \sin\theta = -\sqrt{\frac{3}{8\pi}}\,[x+iy]$$

$$Y_{10} = \frac{1}{\sqrt{2}} e^{-i\phi} \left(i(\cos\theta)\, i - \cos\theta\right)\left(-\sqrt{\frac{3}{8\pi}} e^{i\phi}\right)$$

$$= \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\,[z]$$

$$Y_{1-1} = \frac{1}{\sqrt{2}} e^{-i\phi} \cdot \sqrt{\frac{3}{4\pi}} \sin\theta = \sqrt{\frac{3}{8\pi}}\, e^{-i\phi} \sin\theta = \sqrt{\frac{3}{8\pi}}\,[x-iy]$$

$\ell = 2$: Homework.

Functions on S^2 spanned by $|\ell, m\rangle$,

$$Y_{\ell m}(\theta,\phi) = \langle \theta, \phi | \ell, m\rangle$$

Completeness:
$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') = \frac{\delta(\theta-\theta')\,\delta(\phi-\phi')}{\sin\theta}$$

$$\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

<u>Application of spherical harmonics</u>: separation of variables.

If $V(r)$ is spherically symmetric, $H\psi = E\psi$

for $H = \frac{p^2}{2m} + V(r)$

solutions are of form $\psi_{E\ell m} = \frac{u_{E\ell}(r)}{r} Y_{\ell m}(\theta,\phi)$

where $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \left[\frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right]\right] u_{E\ell}(r) = E\, u_{E\ell}(r)$

— reduces to 1D problem with new potential (HW)

3.4 Addition of angular momenta

Reducible representations

A representation $\mathcal{D}$ of an algebra $\mathfrak{g}$, $\mathcal{D}(K): \mathcal{H} \to \mathcal{H}$ $\forall K \in \mathfrak{g}$ is reducible if

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where

$$\begin{aligned} \mathcal{D}(K) &= \mathcal{D}_1(K) \oplus \mathcal{D}_2(K), \\ \mathcal{D}_1(K) &: \mathcal{H}_1 \to \mathcal{H}_1 \\ \mathcal{D}_2(K) &: \mathcal{H}_2 \to \mathcal{H}_2 \end{aligned} \qquad \forall K \in \mathfrak{g}$$

i.e., $\mathcal{D}(K)$ is block-diagonal $\forall K$.

$$\left(\begin{array}{c|c} \mathcal{D}_1 & 0 \\ \hline 0 & \mathcal{D}_2 \end{array}\right)$$

The representation is irreducible if this is not possible.

Spin j reps are all irreducible representations of $SU(2)$. Any other representation is a direct sum of irreps.

$$\mathcal{H} = \mathcal{H}_{j_1} \oplus \mathcal{H}_{j_2} \oplus \mathcal{H}_{j_3} \oplus \cdots$$

Question: Given two systems, one ($\mathcal{H}_1$) with spin j_1, the other ($\mathcal{H}_2$) with spin j_2, how can we classify angular momentum of the combined system [recall tensor product spaces]

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

One basis: $|m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$.

Total angular momentum is given by

$$J_i = J_i^{(1)} + J_i^{(2)} \qquad \left[= J_i^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J_i^{(2)} \right]$$

$$J^2 = J_{(1)}^2 + J_{(2)}^2 + 2\vec{J}_{(1)} \cdot \vec{J}_{(2)}.$$

Now, $\quad [J^2, J_z^{(1)}] \neq 0,$

so J^2 not a good quantum number in basis $|j_1, m_1; j_2, m_2\rangle$

For total system, want to diagonalize J^2, J_z.
use total j, m as quantum numbers.

What are possible values of j, m given j_1, j_2?

<u>Example</u>: two spin-1/2 particles ($j_1 = j_2 = 1/2$)

States & J_z eigenvalues $\quad (m = m_1 + m_2)$

states	m
$\|++\rangle$	1
$\|+-\rangle \quad \|-+\rangle$	0
$\|--\rangle$	-1

Clearly, quantum numbers are those of

one spin-1 multiplet $\quad (m = -1, 0, 1)$
one spin ϕ multiplet $\quad (m = 0)$

So another basis is $|j, m\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle, \quad |0,0\rangle$.

What are coefficients for a change of basis

$$\langle j, m \mid m_1, m_2 \rangle \quad \left(\begin{array}{l} \text{given } j_1, j_2: \quad \text{often written as} \\ \langle j_1 j_1, j_2, m \mid j_1, m_1; j_2, m_2 \rangle \end{array} \right)$$

or

$$\langle j_1 j_2; j, m \mid j_1, j_2; m_1, m_2 \rangle$$

(book)

Clebsch-Gordon coefficients

Can calculate by recursion, using J_-.

Clearly $\boxed{|j=1, m=1\rangle = |++\rangle}$ (up to sign)

$$J_- |1,1\rangle = \hbar\sqrt{2}\, |1,0\rangle$$
$$= (J_-^{(1)} + J_-^{(2)}) |++\rangle$$
$$= \hbar (|+-\rangle + |-+\rangle)$$

so $\boxed{|1,0\rangle = 1/\sqrt{2}\,(|+-\rangle + |-+\rangle)}$

$$J_- |1,0\rangle = \hbar\sqrt{2}\, |1,-1\rangle$$
$$= (J_-^{(1)} + J_-^{(2)}) \tfrac{1}{\sqrt{2}} (|+,-\rangle + |-,+\rangle)$$
$$= \hbar\sqrt{2}\, |--\rangle$$

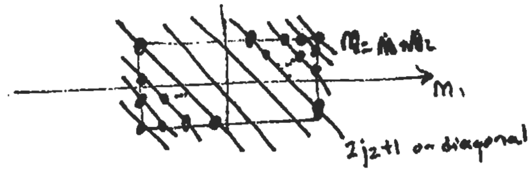
so $\boxed{|1,-1\rangle = |--\rangle}$.

By orthogonality,

$$|0,0\rangle = \tfrac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) \quad \left[\begin{array}{l}\text{up to conventional} \\ \text{sign / phase}\end{array}\right]$$

Check: $J_z |0,0\rangle = 0$

$$J^2 |0,0\rangle = (J_{(1)}^2 + J_{(2)}^2 + 2\vec{J}_{(1)} \cdot \vec{J}_{(2)}) \tfrac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle)$$
$$= \left(3\left[\tfrac{1}{4}\hbar^2 + \tfrac{1}{4}\hbar^2 - \tfrac{1}{2}\hbar^2\right] \right) \tfrac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) = 0.$$



Generally, add spin j_1, spin j_2 — assume $j_1 \geq j_2$ wlog.
Diagonalize in $m = m_1 + m_2$

m	# states	States
$j_1 + j_2$	1	$\|m_1 = j_1, m_2 = j_2\rangle$
$j_1 + j_2 - 1$	2	$\|j_1, j_2 - 1\rangle, \|j_1 - 1, j_2\rangle$
$j_1 + j_2 - 2$	3	$\|j_1, j_2 - 2\rangle, \|j_1 - 1, j_2 - 1\rangle, \|j_1 - 2, j_2\rangle$
$\vdots$		
$j_1 - j_2$	$2j_2 + 1$	$\|j_1, -j_2\rangle, \ldots \|j_1 - 2j_2, j_2\rangle$
$\vdots$	$(2j_2 + 1)$	
$j_2 - j_1$	$2j_2 + 1$	$\|2j_2 - j_1, -j_2\rangle \cdots \|-j_1, j_2\rangle$
$j_2 - j_1 - 1$	$2j_2$	$\|2j_2 - j_1 - 1, -j_2\rangle \cdots \|-j_1, j_2 - 1\rangle$
$\vdots$		$\vdots$
$-j_1 - j_2 + 1$	2	$\|-j_1 + 1, -j_2\rangle \quad \|-j_1, -j_2 + 1\rangle$
$-j_1 - j_2$	1	$\|-j_1, -j_2\rangle$

Gives all states associated with
one spin-j multiplet for each j: $|j_1 - j_2| \leq j \leq j_1 + j_2$

Counting # of states ($j_1 \geq j_2$)

$$\sum_{j = j_1 - j_2}^{j_1 + j_2} 2j+1 = (j_1 + j_2 + 1)^2 - (j_1 - j_2)^2$$

$$= (2j_1 + 1)(2j_2 + 1) \quad \checkmark.$$

Can calculate all Clebsch's $\langle j, m | j_1, m_1; j_2, m_2\rangle$
using J_-'s recursively as before.

First set $|j = j_1 + j_2, m = j_1 + j_2\rangle = |j_1, m_1 = j_1; j_2, m_2 = j_2\rangle$
Construct $|j_1 + j_2, m\rangle$ using J_-,
$|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$ using orthog.
$|j_1 + j_2 - 1, m\rangle$ using J_-, etc...

Generally,

$$\langle j, m | j_1, m_1; j_2, m_2 \rangle = 0$$

unless $m = m_1 + m_2$, $|j_1 - j_2| \le j \le j_1 + j_2$.

Another useful example: $j_1 = \ell$, $j_2 = 1/2$

(spin-1/2 particle with orbital angular momentum)

Expect

$$|j = \ell + 1/2, m = 1/2 + m_1\rangle = \alpha\, |m_1, 1/2\rangle + \beta\, |m_1+1, -1/2\rangle$$

act with $J^2/\hbar^2$

$$(\ell + 1/2)(\ell + 3/2)\left[\alpha |m_1, 1/2\rangle + \beta |m_1+1, -1/2\rangle\right]$$

$$= \left(L^2 + S^2 + 2L_zS_z + L_+S_- + L_-S_+\right)\left[\alpha |m_1, 1/2\rangle + \beta |m_1+1, -1/2\rangle\right)$$

$$= \left[\alpha\left[\ell(\ell+1) + \tfrac{3}{4} + m_1\right] + \beta\sqrt{\ell(\ell+1) - m_1(m_1+1)}\right]|m_1, 1/2\rangle$$
$$+ \left[\quad \cdots \quad\right]|m_1+1, -1/2\rangle$$

$$\Rightarrow \quad \alpha(\ell - m_1) = \beta\sqrt{(\ell+m_1+1)(\ell-m)} \qquad \left[\text{from } |m_1, 1/2\rangle \text{ or } |m_1+1, -1/2\rangle\right]$$

so $$\frac{\alpha}{\beta} = \sqrt{\frac{\ell+m_1+1}{\ell-m_1}}$$

Normalization: $\alpha^2 + \beta^2 = 1 \Rightarrow \alpha = \sqrt{\frac{\ell+m_1+1}{2\ell+1}}$ $\beta = \sqrt{\frac{\ell-m_1}{2\ell+1}}$

so $$|j = \ell + 1/2, m = m_1 + 1/2\rangle = \sqrt{\frac{\ell+m_1+1}{2\ell+1}}\,|m_1, 1/2\rangle + \sqrt{\frac{\ell-m}{2\ell+1}}\,|m+1, -1/2\rangle$$
$$= \sqrt{\frac{\ell+m_1+1}{2\ell+1}}\,Y_{\ell m}|+\rangle + \sqrt{\frac{\ell-m_1}{2\ell+1}}\,Y_{\ell m+1}|-\rangle$$

$$|j = \ell - 1/2, m = m_1 + 1/2\rangle = -\sqrt{\frac{\ell+m_1+1}{2\ell+1}}\,|m_1+1, -1/2\rangle + \sqrt{\frac{\ell-m_1}{2\ell+1}}\,|m_1, +1/2\rangle$$

by orthogonality.

Last time: discussed Clebsch-Gordan coefficients $\langle jm | j_1 m_1 ; j_2, m_2 \rangle$

Given $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. CG coefts give transformation between bases

$|j, m\rangle$ eigenvectors of J^2, J_z

$|j_1 m_1 ; j_2 m_2\rangle$ eigenvectors of $J_z^{(1)}, J_z^{(2)}$.

$$\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)} = \mathcal{D}^{(j_1+j_2)} \oplus \mathcal{D}^{(j_1+j_2-1)} \oplus \cdots \oplus \mathcal{D}^{(|j_1-j_2|)}$$

$$\mathcal{D}^{(j_1)}_{m_1 m_1'}(R)\mathcal{D}^{(j_2)}_{m_2 m_2'}(R) = \langle j_1 m_1 ; j_2 m_2 | \mathcal{D}(R) | j_1, m_1' ; j_2, m_2' \rangle$$

$$= \sum_{j, m, m'} \langle j_1 m_1 ; j_2 m_2 | j, m\rangle \mathcal{D}^{(j)}_{m m'} \langle j, m' | j_1, m_1' ; j_2, m_2' \rangle \quad (*)$$

Showed how to compute $\langle j, m | j_1 m_1 ; j_2 m_2 \rangle$ recursively.

Closed form expression (Racah, etc)

$$\langle j_1 m_1 ; j_2 m_2 | j, m \rangle = \delta_{m_1+m_2, m} \sqrt{2j+1} \left[\frac{(j_1+j_2-j)!\,(j_1-j_2+j)!\,(-j_1+j_2+j)!}{(j_1+j_2+j+1)!} \right]^{1/2} \times$$

$$\left[(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!\right]^{1/2} \times$$

$$\sum_n \frac{1}{n!\,(j_1+j_2-j-n)!\,(j_1-m_1-n)!\,(j_2+m_2-n)!\,(j-j_2+m_1+n)!\,(j-j_1-m_2+n)!}$$

(Sum over all integers n s.t. all !'s are nonnegative.

Note: all CG's are real

Note: symm under perm of $(j, m), (j_1 m_1), (j_2, m_2)$ up to sign

$$= \sqrt{2j+1}\,(-1)^{j_1+j_2-m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \quad \text{"3j" symbol}$$

3.5 Tensor operators & the Wigner-Eckart theorem

So far we have discussed how states xform under rotations.
Now useful to discuss " operators " " ".

Classically, a vector V^i transforms under rotations as

$$V^i \to R^i{}_j V^j \quad \text{(summation convention)} \qquad \text{(Ex. position } r^i\text{)}$$

An n-tensor $T^{i_1 \dots i_n}$ transforms as

$$T^{i_1 \dots i_n} \to R^{i_1}{}_{j_1} R^{i_2}{}_{j_2} \cdots R^{i_n}{}_{j_n} T^{j_1 \dots j_n}$$

Ex. dyadic $r^i r^j$, as in inertia tensor

$$A^{ij} = \int d^3r \, \rho(r) \left[r^2 \delta^{ij} - r^i r^j \right]$$

Expect similar behavior for quantum operators, expectation values

$$|\alpha\rangle \to \mathcal{D}(R) |\alpha\rangle$$

$$\langle\alpha| V^i |\alpha\rangle \to \langle\alpha| \mathcal{D}^\dagger(R) V^i \mathcal{D}(R) |\alpha\rangle$$
$$= R^i{}_j \langle\alpha| V^j |\alpha\rangle$$

for any $|\alpha\rangle$ $\Rightarrow$ $\boxed{\mathcal{D}^\dagger(R) V^i \mathcal{D}(R) = R^i{}_j V^j.}$

this equation defines a <u>vector operator</u>

Infinitesimal version:

$$\boxed{[V^i, J^j] = i\hbar\varepsilon^{ijk} V_k}$$

(no matter what representation J is in.)

Exs. $V^i = x^i,\ p_i$

[From $L_z = xp_y - yp_x$, $\quad [x, L_z] = -i\hbar y$
$\qquad [p_x, L_z] = -i\hbar p_y, \ldots$]

For higher rank tensors more complicated.

For example, dyadic U^iV^j (cartesian tensor) has 3 parts (xforms as $\mathcal{H}_{j=1} \otimes \mathcal{H}_{j=1}$)

$(U\cdot V)\,\delta^{ij}$ xforms as scalar (spin 0) – 1 component

$U^iV^j - U^jV^i$ xforms as vector $U\times V$ (spin 1) – 3 components

$\frac{1}{2}(U^iV^j + U^jV^i) - \frac{1}{3}U\cdot V\,\delta^{ij}$ xforms as symmetric traceless mtx (spin 2) – 5 components

Can see decomposition of $j_1 = 1,\ j_2 = 1 \to j = 0, 1, 2$
$3\cdot 3 = 9 = 1 + 3 + 5.$

More generally, <u>spherical tensors</u>

Irreducible spherical tensor $T_q^{(k)}$ is a tensor operator of rank k with $2k+1$ components $q = -k, -k+1, \ldots, k$ such that

$\mathcal{D}$'s acting on $\mathcal{H}$

$$\mathcal{D}(R^{-1})\, T_q^{(k)}\, \mathcal{D}(R) = \sum_{q'=-k}^{k} \mathcal{D}_{qq'}^{(k)*}(R)\, T_{q'}^{(k)}$$

$\downarrow R \to R^{-1}$

$$\mathcal{D}(R)\, T_q^{(k)}\, \mathcal{D}^\dagger(R) = \sum_{q'=-k}^{k} \mathcal{D}_{qq'}^{(k)}(R)\, T_{q'}^{(k)}$$

Infinitesimally,

$$[J_z, T_q^{(k)}] = \hbar q\, T_q^{(k)}$$
$$[J_\pm, T_q^{k}] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\; T_{q\pm 1}^{(k)}$$

Exs. explicit construction of spherical tensors from vector operators

Recall $Y_{\ell m}$ is a function of $\hat{n} = (x, y, z) \in S^2 = \{(x,y,z) : x^2+y^2+z^2=1\}$

Take $Y_{\ell m}(x,y,z) \longrightarrow T_q^{(k)} = Y_{kq}(V_x, V_y, V_z)$

gives spherical tensor

Ex.
$$T_0^{(1)} = \sqrt{\frac{3}{4\pi}}\, V_z$$
$$T_{\pm 1}^{(1)} = \sqrt{\frac{3}{4\pi}}\left(\mp \frac{V_x \pm i V_y}{\sqrt{2}}\right).$$

Spherical tensors combine just like kets to form higher-spin & lower-spin tensors.

Theorem: If $X_{q_1}^{(k_1)}, Z_{q_2}^{(k_2)}$ are irreducible spherical tensors of rank k_1, k_2 then

$$T_q^{(k)} = \sum_{q_1, q_2} \langle k_1, q_1; k_2, q_2 | k, q\rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

is an irreducible rank k spherical tensor.

<u>Proof</u>

$$\mathcal{D}(R)\, T_q^{(k)}\, \mathcal{D}^\dagger(R) = \sum_{q_1 q_2} \langle k_1, q_1; k_2, q_2 | k, q\rangle \underbrace{\left[\mathcal{D}(R) X_{q_1}^{(k_1)} \mathcal{D}^\dagger(R)\right]}_{\mathcal{D}_{q_1 p_1}^{(k_1)} X_{p_1}^{(k_1)}} \underbrace{\left[\mathcal{D}(R) Z_{q_2}^{(k_2)} \mathcal{D}^\dagger(R)\right]}_{\mathcal{D}_{q_2 p_2}^{(k_2)} Z_{p_2}^{(k_2)}}$$

$$= \sum_p \mathcal{D}_{qp}^{(k)} \langle k_1, p_1; k_2, p_2 | k, p\rangle X_{p_1}^{(k_1)} Z_{p_2}^{(k_2)}$$

$$= \sum_p \mathcal{D}_{qp}^{(k)} T_p^{(k)} \quad \square$$

<u>Wigner - Eckart</u>

Often useful to calculate matrix elements of spherical tensors between angular momentum eigenstates

$$\langle \alpha'; j', m' | T_q^{(k)} | \alpha; j, m\rangle$$

↑ (other q. numbers besides A.M.)

(For example, for coupling to EM field, radiation, etc ...)

Wigner - Eckart:

$$\langle \alpha'; j', m' | T_q^{(k)} | \alpha; j, m\rangle = \overbrace{\langle j', m' | k, q; j, m\rangle}^{\text{only dependence on geometry}} \frac{\langle \alpha', j' \| T^{(k)} \| \alpha, j\rangle}{\sqrt{2j+1}}$$

where $\langle \alpha', j' \| T^{(k)} \| \alpha, j\rangle$ is independent of m, q, m'.

<u>Proof</u>: Note recursion relation

$$\langle j, m | J_\pm | j_1, m_1; j_2, m_2\rangle$$

$$= \sqrt{(j \pm m)(j \mp m + 1)}\, \langle j, m \mp 1 | j_1, m_1; j_2, m_2\rangle$$

$$= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\, \langle j, m | j_1, m_1 \pm 1; j_2, m_2\rangle$$
$$+ \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\, \langle j, m | j_1, m_1; j_2, m_2 \pm 1\rangle$$

Same relation for mtx elements of $T^{(k)}_q$:

$$\langle \alpha'; j', m' | J_\pm T^{(k)}_q | \alpha, j, m\rangle$$
$$= \sqrt{(j' \pm m')(j' \mp m'+1)}\ \langle \alpha'; j', m' \mp 1 | T^{(k)}_q | \alpha; j, m\rangle$$
$$= \sqrt{(k\mp q)(k \pm q+1)}\ \langle \alpha'; j', m' | T^{(k)}_{q\pm 1} | \alpha; j, m\rangle$$
$$+ \sqrt{(j\mp m)(j \pm m+1)}\ \langle \alpha'; j', m' | T^{(k)}_{q} | \alpha; j, m\pm 1\rangle$$

So theorem holds — solution of recursion eqns unique up to a constant for each j, j', k.

<u>Selection rules</u>:

$$m' = q + m.$$

$$|j - q| \le j' \le j + q.$$

[tells us what kind of radiation from certain emissions, etc.]

<u>Examples</u>:

$k = 0$: scalar operator $S = T^{(0)}_0$

$$\langle \alpha'; j', m' | S | \alpha; j, m\rangle = \delta_{jj'} \delta_{mm'} \frac{\langle \alpha' j' \| S \| \alpha j\rangle}{\sqrt{2j+1}}$$

S cannot change j, m.

$K=1$: Vector operator $V_0, V_{\pm 1}$

selection rules: $\Delta m = m' - m = 0, \pm 1$

$\Delta j = j' - j = 0, \pm 1$, can't have $j = j' = 0$.

$j = j'$, $K = 1$:

Projection theorem

$$\langle \alpha'; j, m' | V_q | \alpha; j, m \rangle = \frac{\langle \alpha'; j, m | J \cdot V | \alpha; j, m \rangle}{\hbar^2 j(j+1)} \langle j, m' | J_q | j, m \rangle$$

where $V_{\pm 1} = \mp \frac{1}{\sqrt{2}} (V_x \pm i V_y)$, $V_0 = V_z$

$J_{\pm 1} = \mp \frac{1}{\sqrt{2}} (J_x \pm i J_y) = \mp \frac{1}{\sqrt{2}} J_{\pm}$ $\quad J_0 = J_z$.

Proof

$$\langle \alpha'; j, m | J \cdot V | \alpha; j, m \rangle = \langle \alpha'; j, m | (J_0 V_0 + J_+ V_- + J_- V_+) | \alpha; j, m \rangle$$

$$= m\hbar \langle \alpha'; j, m | V_0 | \alpha; j, m \rangle$$

$$+ \frac{\hbar}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \langle \alpha'; j, m-1 | V_{-1} | \alpha; j, m \rangle$$

$$- \frac{\hbar}{\sqrt{2}} \sqrt{(j-m)(j+m+1)} \langle \alpha'; j, m+1 | V_{+1} | \alpha; j, m \rangle$$

$$= C_j \langle \alpha' j \| V \| \alpha j \rangle \quad \text{by W.E.}$$

C_j independent of V, α, α', log. distinct & m since $J \cdot V$ scalar.

choose $V = J, \alpha = \alpha'$

$$\langle \alpha ; j, m | J^2 | \alpha ; j, m \rangle = C_j \langle \alpha' ; j \| J \| \alpha j \rangle$$

$$= \hbar^2 j(j+1)$$

but $$\frac{\langle \alpha' ; j, m' | V_q | \alpha ; j, m \rangle}{\langle \alpha ; j, m' | J_q | \alpha ; j, m \rangle} = \frac{\langle \alpha' ; j \| V \| \alpha j \rangle}{\langle \alpha j \| J \| \alpha j \rangle}$$

so

$$\langle \alpha' ; j, m' | V_q | \alpha ; j, m \rangle = \langle j, m' | J_q | j, m \rangle \frac{\langle \alpha' ; j, m | J \cdot V | \alpha ; j, m \rangle}{\hbar^2 j(j+1)}$$

$\square$