Lecture 12 (Oct. 23, 2017)

12.1 Stationary Phase Approximation

Consider an integral of the form

$$\int \mathrm{d}x\, e^{i\lambda f(x)}\,, \tag{12.1}$$

with λ large and $f(x)$ real. Can we use the largeness of λ in order to help evaluate or approximate the integral? Let $f(x)$ have an extremum at $x = x_0$. The claim is that the integral will be dominated by the region around x_0. In most regions of integration, the complex exponential oscillates rapidly as a function of x, and so the integral over any range will sum many phases that cancel out. The region around x_0, however, varies slowly, because $f(x)$ achieves an extremum at x_0. We thus expect that the integral is dominated by x near x_0 (or, in general, near all x_0 for which $f(x)$ achieves an extremum).

More precisely, near x_0, we can write

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cdots\,, \tag{12.2}$$

where the first-order term vanishes because $f(x)$ has an extremum at x_0. We can then write

$$\int \mathrm{d}x\, e^{i\lambda f(x)} \approx e^{i\lambda f(x_0)} \int \mathrm{d}(\delta x)\, e^{i\lambda f''(x_0)(\delta x)^2/2+\cdots}\,. \tag{12.3}$$

If we assume that δx is small, we can drop the higher order terms in the exponential. However, we can then extend the limits of integration over δx to $\pm\infty$, because the largeness of λ leads to rapid oscillation for large δx, giving cancellation of the exponentials. This yields

$$\begin{aligned}\int \mathrm{d}x\, e^{i\lambda f(x)} &\approx e^{i\lambda f(x_0)} \int_{-\infty}^{\infty} \mathrm{d}(\delta x)\, e^{i\lambda f''(x_0)(\delta x)^2/2} \\ &= e^{i\lambda f(x)} \sqrt{\frac{2\pi i}{\lambda f''(x_0)}} \big(1 + O(\lambda^{-2})\big)\,.\end{aligned} \tag{12.4}$$

We can generalize this to multidimensional integrals, and even path integrals. We thus similarly expect that

$$\int [\mathcal{D}x]\, e^{iS[x(t)]/\hbar} \tag{12.5}$$

is dominated by $x(t) = x_{\mathrm{cl}}(t)$, the classical path, where

$$\left.\frac{\delta S}{\delta x}\right|_{x_{\mathrm{cl}}} = 0\,, \tag{12.6}$$

when $\hbar \to 0$. Then we get

$$K(x_f, t_f; x_i, t_i) \sim e^{iS[x_{\mathrm{cl}}(t)]/\hbar}\,. \tag{12.7}$$

Note that this not only gives us the classical limit, but also gives us interference effects in the near-classical limit: if there are multiple possible classical paths, then in the near-classical limit, these paths can exhibit interference with one another.

12.2 Quantum Particles in Electromagnetic fields

We now discuss the motion of quantum particles in electromagnetic fields. In general, we should assume that every system is quantum mechanical. However, here we will be using a semiclassical approach, in which we will consider a classical background electromagnetic field with quantum particles moving in it. There are many phenomena which require the quantization of the electric field as well as the particles moving in it, but we will not discuss such phenomena here.

12.2.1 Constant Potentials

We start with a potential V. If we change the potential by $V \to V + V_0$, this is a shift in the overall energy, and should have no effect on the particle motion. Classically, this is true because particles move due to forces, not potentials. The force is computed by taking the gradient of the potential, which will not be effected by the shift. In quantum mechanics, the shift in the potential results in an extra phase under time evolution,

$$|\psi(t)\rangle \to e^{-iV_0 t/\hbar}|\psi(t)\rangle\,. \tag{12.8}$$

Thus, the shift serves to change the phase of the wavefunction at each time step. We know that such an overall phase shift is not observable.

However, if we change the potential by a constant shift in one region of space but not in another, then we do expect to see an observable effect. Quantum mechanically, we can interfere particles that have traveled through this region with those that have not.

Imagine that we have a beam of particles being emitted from a source A, and two metallic cages that the beam could pass through, before passing to a detector at B. Assume that the potential is V_1 in the bottom cage and V_2 in the top cage, both constant. The potential is constant within each cage, and so the particles passing through either cage experience no forces. Quantum mechanically, there is nevertheless a difference in the phase factor accumulated while passing through the two different cages:

$$\phi_2 - \phi_1 = \frac{1}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\, \left(V_2(t) - V_1(t)\right). \tag{12.9}$$

Let us assume that the two metallic cages are completely identical other than the described potential difference, and that A and B are arranged symmetrically around the two cages. We thus expect, from symmetry, that the amplitude for a particle to travel from A to B would be the same through either cage in absence of the potential difference. If we let a be the amplitude to go from A to B through the bottom cage in time $t_f - t_i$, then the amplitude to go through the top cage must be

$$ae^{-i(\phi_2-\phi_1)}\,. \tag{12.10}$$

The full amplitude to travel from A to B is then

$$a\Big(1 + e^{-i(\phi_2-\phi_1)}\Big)\,, \tag{12.11}$$

giving a probability of detection of

$$\mathrm{Prob}(A \to B) = 2|a|^2(1 + \cos(\phi_2 - \phi_1))\,. \tag{12.12}$$

Thus, the constant potential difference between the two cages changes physical observables; this is a purely quantum mechanical effect, as the particles experienced no forces during this experiment (neglecting the small forces when entering and exiting the regions of shifted potential). Note that in the limit of $\hbar \to 0$, the phase difference (12.9) oscillates very rapidly. When averaged over any nonzero time interval, these rapidly oscillating phases cancel out, so the interference effects will disappear, as we expect in the classical limit.

12.2.2 Electromagnetic Fields

Let us briefly review electromagnetism. We know that we can represent electric fields $\boldsymbol{E}$ and magnetic fields $\boldsymbol{B}$ through scalar and vector potentials as

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\,, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}\,. \tag{12.13}$$

Note that we are using cgs units. The potentials ϕ and $\boldsymbol{A}$ will be extremely useful in describing the motion of quantum mechanical particles through EM fields.

It is useful to introduce a more formal notation, defining the electromagnetic field strength tensor

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{12.14}$$

with

$$A_\mu = (\phi, \boldsymbol{A})\,, \quad (\mu = 0, 1, 2, 3)\,. \tag{12.15}$$

The electric and magnetic fields are then given by

$$F_{0i} = -F_{i0} = -E_i\,, \quad F_{ij} = \epsilon_{ijk} B_k\,. \tag{12.16}$$

The four-vector potential A_μ is redundant: the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu A \tag{12.17}$$

leaves the field strength $F_{\mu\nu}$ invariant, which means the electric and magnetic fields are unchanged. The field strength tensor is the real physical quantity, and all of the redundant values of A_μ that yield the same field strength tensor describe the same physics.

For a charged particle in an electromagnetic field, the Lagrangian is

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + \frac{e}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{x}} - e\phi\,. \tag{12.18}$$

In Lagrangian mechanics, which is formulated in terms of position and velocity, there is a notion of momentum as the variable conjugate to the velocity. The *canonical momentum* is given by

$$\boldsymbol{p} := \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}\,. \tag{12.19}$$

Note that the canonical momentum is *not* gauge-invariant; it depends on the choice of gauge.

Using this canonical momentum, we find that the Hamiltonian is given by

$$H = \boldsymbol{p}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + e\phi\,. \tag{12.20}$$

This is the expected answer; the Hamiltonian is gauge-invariant. Using the definition of the canonical momentum, we can rewrite this as

$$H = \frac{\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2}{2m} + e\phi\,. \tag{12.21}$$

The main conceptual point here is that the *canonical* momentum $\boldsymbol{p}$ is different from $m\boldsymbol{v} = m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$ (the "kinematic" momentum) when $\boldsymbol{A} \neq 0$. The reason we care about the canonical momentum rather than the kinematic momentum is because it is the quantity that has $\{x, p\} = 1$ (with $\{\cdot, \cdot\}$

the Poisson bracket), which means that it will have the proper commutator with the operator $\boldsymbol{x}$ when we quantize the theory.

Quantum mechanically, we use the Hamiltonian (12.21) to discuss quantum motion, using the commutators

$$[x_i, p_j] = i\hbar\delta_{ij}\,, \quad [x_i, x_j] = [p_i, p_j] = 0\,. \tag{12.22}$$

Expanding Eq. (12.21), we have

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{e}{2mc}(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}) + \frac{e^2}{2mc^2}\boldsymbol{A}^2 + e\phi\,. \tag{12.23}$$

Note that the ordering of $\boldsymbol{p}$ and $\boldsymbol{A}$ matters here, as they may not commute.

12.2.3 Gauge Invariance in Quantum Mechanics

The Schrödinger equation is

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= \left[\frac{\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2}{2m} + e\phi\right]\psi \\ &= \left[\frac{\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)^2}{2m} + e\phi\right]\psi\,. \end{aligned} \tag{12.24}$$

How should we change the wavefunction so that the theory remains invariant under gauge transformations?

Let

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda\,, \quad \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\,, \tag{12.25}$$

which is a gauge transformation, where $\Lambda(\boldsymbol{x}, t)$ is a scalar function. Then we have

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= \left[\frac{\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2}{2m} + e\phi\right]\psi \\ &= \left[\frac{\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}' + \frac{e}{c}\nabla\Lambda\right)^2}{2m} + e\phi' + \frac{e}{c}\frac{\partial\Lambda}{\partial t}\right]\psi\,. \end{aligned} \tag{12.26}$$

We can rearrange this equation to reach

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi' - \frac{e}{c}\frac{\partial\Lambda}{\partial t}\right)\psi = \frac{\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}' + \frac{e}{c}\nabla\Lambda\right)^2}{2m}\psi\,. \tag{12.27}$$

We can compensate the changes from the gauge transformation by letting

$$\psi = \psi' e^{-ie\Lambda/\hbar c}\,, \tag{12.28}$$

which leaves us with

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi'\right)\psi' = \frac{\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}'\right)^2}{2m}\psi'\,, \tag{12.29}$$

matching the form of the Schrödinger equation in Eq. (12.24).

Thus, the full gauge transformation is

$$\begin{aligned} \boldsymbol{A}' &= \boldsymbol{A} + \nabla\Lambda\,, \\ \phi' &= \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\,, \\ \psi &= \psi' e^{-ie\Lambda/\hbar c}\,. \end{aligned} \tag{12.30}$$

All observables must remain invariant under this transformation.

12.2.4 Aharonov–Bohm Effect

Consider a hollow cylindrical shell with a solenoid through the middle, such that there is a flux

$$\Phi = \int_{\text{core}} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} \tag{12.31}$$

through the core, but nowhere in the shell. Suppose that some charged particles are constrained to move within the shell, where there is no magnetic field. Classically, the magnetic field does not affect the motion of the charged particles, because they experience no forces. What about in quantum mechanics?

In order to formulate the quantum motion, we need to determine the vector potential $\boldsymbol{A}$ (note that the scalar potential is zero in this case). Even though the magnetic field is zero in the shell where the particles move, the vector potential cannot be chosen to vanish in this region. To see this, consider taking the line integral of $\boldsymbol{A}$ around a closed contour C lying within the shell and enclosing the core. Using Stokes' theorem, we see that this line integral is given by

$$\oint_C \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{\ell} = \int_\Sigma (\nabla\times\boldsymbol{A})\cdot \mathrm{d}\boldsymbol{S} = \int_\Sigma \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \Phi\,, \tag{12.32}$$

where Σ is the area bounded by the contour C, i.e., $C = \partial\Sigma$. Because there is nonzero flux through the core, this integral cannot be zero. Thus, we cannot set $\boldsymbol{A}$ to zero everywhere in the shell.