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PROFESSOR: OK. Let's start. So let's review what we've been trying to do in the past few lectures. It's to understand a typical situation involving something like a gas, which, let's say, initially is in one half of the container. And then expands until it occupies the entire container, going from one equilibrium state to another equilibrium state.

Question is, how to describe this from the perspective of microscopic degrees of freedom, getting the idea that eventually you will come to some form of equilibrium. And actually, more seriously, quantifying the process and timescales by which the gas comes to equilibrium.

So the first stage was to say that we will describe the process by looking at the density, which describes the number of representative examples of the system, whose collections of coordinates and momenta occupy a particular point in the $6N$ -dimensional phase space. So basically, this is a function that depends on $6N$ coordinates. And of course, it changes as a function of time.

And we saw that, basically, if we were to look at how this set of points were streaming in phase space, and this can be mathematically represented by looking at the direct time dependence and the implicit time dependence of all of the coordinates captured through the Poisson bracket. And the answer was that it is a characteristic of the type of Hamiltonian evolution equations, that the flow is such that the phase space does not change volume. And hence, this quantity is 0.

We said, OK, all of that is fine. But what does that tell me about how this gas is expanding? So for that, we said that we need to look at descriptions that are more appropriate to thinking about the volume and density of a gas expanding. And we really want to capture that through the one-particle density.

But along the way, we introduced s particle densities, normalized up to something, to correspond to, essentially, integrating over all of the coordinates that we're not interested in. Say coordinates pertaining to numbers $s + 1$ to N , corresponding $6N$ -dimensional phase space and the entire [INAUDIBLE] of this density or probability.

Then the next stage was to be more specific about what governs the evolution of these gas particles, i.e., what is the Hamiltonian. And we said, let's focus on an N particle Hamiltonian that is composed, by necessity certainly, of one-body terms that is the kinetic energy and the potential energy, the latter describing the box, for example. And we added a two-particle interaction. q_i minus q_j , which, for the sake of simplicity, let's imagine is spherically symmetric-- only depends on the relative position of these particles.

Then we say that if I look at the evolution of this s particle density, I start by writing an equation that kind of looks like the full Liouville equation. There is a Hamiltonian that describes this set of particles interacting among themselves, just replacing n by s . And so if these were the only particles in the universe, this is what I would get.

Except that we know that there are other particles. And since we have two-body collisions possible, any one of the particles in the set that I'm describing here, composed of s particles, can potentially collide with a particle from this set that I don't want to include. And that will change the volume of phase space. This incompressibility condition was valid only if the entire thing of the Hamiltonian was considered. Here, I'm only looking at the partial subset.

And so then what did I have here? I had the force exerted from particle $s + 1$ on particle n changing the momentum of the particle n . But the likelihood that this happens depends on finding the other particle also. So I had this dependence on the density that was $s + 1$. So this was the BBGKY hierarchy.

Then we said, let's take a look at the typical value of the terms that we have in this equation. And for s that is larger than 2, this Hamiltonian that pertains to s particles will have a collision term among the particles. And that collision term will give you

some typical inverse timescale here, which is related to the collision time.

Whereas on the right-hand side, for this collision to take place, you have to find another particle to interact with in the range of interactions that we indicated by d .

And for density n , this was smaller by a factor of nd^3 for dilute gases.

Therefore, the left-hand side is much larger than the right-hand side. And within particular approximation, we said that we are going to approximately set this right-hand side to 0. OK?

Except that we couldn't do this for the equation that pertained to s equals to 1, because for s equals to 1, I did not have the analog of this term. If I were to write the precise form of this Poisson bracket, I had the partial derivative with respect to time, the momentum divided by mass, which is the velocity driving the change in coordinate. And the force that comes from the external potential, let's call it f_1 , we arrive in the change in momentum of the one-particle density, which is a function-- let's write it explicitly-- of p_1 , q_1 , and t .

And on the other side, I had the possibility to have a term such as this that describes a collision with a second particle. I have to look at all possible second particles that I can collide with. They could come with all variety of momenta, which we will indicate p_2 . They can be all over the place. So I have integrations over space.

And then I have a bunch of terms. So for this, we said, let's take a slightly better look at the kind of collisions that are implicitly described via the term that we would have here as f_2 . And since the right-hand side of f_2 we set to 0, essentially, f_2 really describes the Newtonian evolution of two particles coming together and having a collision.

And this collision, I can-- really, the way that I have it over there is in what I would call a lab frame. And the perspective that we should have is that in this frame, I have a particle that is coming with momentum, let's say, p_1 , which is the one that I have specified on the left-hand side of the equation.

That's the density that I'm following in the channel represented by p_1 . But then along comes a particle with momentum p_2 . And there is some place where they hit each other, after which this one goes off with p_1 prime. Actually, let me call it p_1 double prime. And this one goes off with p_2 double prime.

Now, we will find it useful to also look at the same thing in the center of mass frame. Now, in the center of mass frame-- oops-- my particle p_1 is coming with a momentum that is shifted by the p of center of mass, which is really p_1 plus p_2 over 2. So this is p_1 minus p_2 over 2. And my particle number two comes with p_2 minus center of mass, which is p_2 minus p_1 over 2. So they're minus each other.

So in the center of mass frame, they are basically coming to each other along the same line. So we could define one of the axes of coordinates as the distance along which they are approaching. And let's say we put the center of mass at the origin, which means that I have two more directions to work with. So essentially, the coordinates of this second particle can be described through a vector b , which we call the impact parameter.

And really, what happens to the collision is classically determined by how close they approach together to make that. To quantify that, we really need this impact vector. So b equals to 0, they are coming at each other head on. For a different b , they are coming at some angle. So really, I need to put here a d^2b .

Now, once I have specified the form of the interaction that is taking place when the two things come together, then the process is completely deterministic. Particle number one will come. And after the location at which you have the interaction, will go off with p_1 double prime minus center of mass. And particle number two will come and get deflected to p_2 prime minus-- or double prime minus p center of mass. OK?

And again, it is important to note that this parameter A is irrelevant to the collision, in that once I have stated that before collision, I have a momentum, say, p_1 and p_2 . And this impact parameter b is specified, then p_1 double prime is known completely as a function of p_1 , p_2 , and b . And p_2 double prime is known completely as a

function of p_1 , p_2 , and b . And everything in that sense is well-defined.

So really, this parameter A is not relevant. And it's really the B that is important. But ultimately, we also need to get these other terms, something that has the appropriate dimensions of inverse time. And if I have a density of particles coming hitting a target, how frequently they hit depends on the product of the density as well as the velocity. So really, I have to add here a term that is related to the velocity with which these things are approaching each other. And so I can write it either as v_2 minus v_1 , or p_2 minus p_1 over m . So that's the other thing that I have to look at.

And then, essentially, what we saw last time is that the rest of the story-- if you sort of think of these as billiard balls, so that the collision is instantaneous and takes place at a well-specified point, instantaneously, at some location, you're going with some momentum, and then turn and go with some other momentum. So the channel that was carrying in momentum p_1 , which is the one that we are interested in, suddenly gets depleted by an amount that is related to the probability of having simultaneously particles of momenta p_1 and p_2 .

And the locations that I have to specify here-- well, again, I'm really following this channel. And in this channel, I have specified q_1 . And suddenly, I see that at the moment of collision, the momentum changes, if you are thinking about billiard balls. So really, it is at that location that I'm interested for particle one.

For particle two, I'm interested at something that is, let's say, q_1 plus, because it depends on the location of particle two shifted from particle one by an amount that is related by b and whatever this amount slightly of A is that is related to the size of your billiard ball. OK?

Now, we said that there is a corresponding addition, which comes from the fact that it is possible suddenly to not have a particle moving along, say, this direction. And the collision of these two particles created something along that direction. So the same way that probabilities can be subtracted amongst some channels, they can be added to channels, because collision of two particles created something. And so I will indicate that by p_1 prime, p_2 prime, again, coordinates in t , something like this.

So basically, I try to be a little bit more careful here, not calling these outgoing channels p prime, but p double prime. Because really, if I were to invert p_1 double prime and p_2 double prime, I will generate things that are going backward.

So there's a couple of sign issues involved. But essentially, there is a similar function that relates p_1 prime and p_2 prime. It is, basically, you have to search out momenta whose outcome will create particles in the channels prescribed by p_1 and p_2 .

So as we said, up to here, we have done a variety of approximations. They're kind of physically reasonable. There is some difficulty here in specifying if I want to be very accurate where the locations of q_1 and q_2 are within this interaction size d that I am looking at.

And if I don't think about billiard balls, where the interactions are not quite instantaneous where you change momenta, but particles that are deformable, there is also certainly a certain amount of time where the potential is acting. And so exactly what time you are looking at here is also not appropriate-- is not completely, no.

But if you sort of make all of those things precise, then my statement is that this equation still respects the reversibility of the equations of motions that we had initially. But at this stage, we say, well, what can I do with this equation? I really want to solve an equation that involves f of p_1 , q_1 , and t .

So the best that I can do is, first of all, simplify all of the arguments here to be more or less at the same location. I don't want to think about one particle being shifted by an amount that is related by b plus something that is related to the interaction potential size, et cetera. Let's sort of change our focus, so that our picture size, essentially, is something that is of the order of the size of the interaction.

And also, our resolution in time is shifted, so that we don't really ask about moments of time slightly before and after collision, where the particle, let's say, could be deformed or whatever. So having done that, we make these approximations that,

essentially, all of these f_2 's, we are going to replace as a product of f_1 's evaluated all at the same location-- the location that is specified here, as well as the same time, the time that is specified there. So this becomes-- OK?

So essentially, the result of the collisions is described on the left-hand side of our equation by a term that integrates over all momenta and overall impact parameters of a bracket, which is a product of two one-particle densities. And just to simplify notation, I will call this the collision term that involves two factors of f_1 . So that's just a shorthand for that. Yes.

AUDIENCE: Is that process equivalent of a similar molecule [INAUDIBLE]?

PROFESSOR: That, if you want to give it a name, yes. It's called the assumption of molecular chaos. And physically, it is the assumption that the probability to find the two particles is the product of one-particle probabilities.

And again, we say that you have to worry about the accuracy of that if you want to focus on sizes that are smaller than the interaction parameter size. But we change that as part of our resolution. So we don't worry about that aspect.

AUDIENCE: So molecular chaos doesn't include eliminating this space? It's a separate thing.

PROFESSOR: I've only seen the word "molecular chaos" applied in this context. I think you are asking whether or not if q_1 and q_2 are different. I can replace that and call that molecular chaos. I would say that that's actually a correct statement. It's not an assumption.

Basically, when the two particles are away from each other to a good approximation, they don't really know about-- well, actually, what I am saying is if I really want to sort of focus on time dependence, maybe that is also an assumption. Yes. So they're really, if you like, indeed, two parts. It is the replacing of f_2 with the product of two f_2 's, and then evaluating them at the same point. Yes.

AUDIENCE: But based on your first line-- I mean, you could break that assumption of different distribution functions have different events that occur when they get close, and the

framework still works, right? It would just become-- when you got down to the bottom line, it would just become a much larger equation, right?

PROFESSOR: I don't understand.

AUDIENCE: Like if you assume different interactions.

PROFESSOR: The interactions are here in how p_1 prime and p_2 prime depend on p_1 and p_2 .

AUDIENCE: Right. What I'm suggesting is if you change H by keeping the top [INAUDIBLE]. That's a pretty simple H . Would this framework work for a more complicated H . But then your bottom two lines would become [INAUDIBLE].

PROFESSOR: What more can I do here? How can I make this more general? Are you thinking about adding it?

AUDIENCE: No, no, no. I'm not suggesting you make it more general. I'm suggesting you make it more specific to break that assumption of molecular chaos.

PROFESSOR: OK. So how should I modify if I'm not saying--

AUDIENCE: I didn't have something in mind. I was just suggesting [INAUDIBLE].

PROFESSOR: Well, that's where I have problem, because it seems to me that this is very general. The only thing that I left out is the body interactions may be having to do with this being spherically symmetric. But apart from that, this is basically as general as you can get with two bodies.

Now, the part that is pertaining to three bodies, again, you would basically-- not need to worry about as long as nd cubed is less than 1. So I don't quite know what you have in mind.

AUDIENCE: I guess I was thinking about three- and four-body interactions. But if you're covered against that, then what I said, it's not valid or relevant.

PROFESSOR: Right. So certainly, three-body and higher body interactions would be important if you're thinking about writing a description for a liquid, for example. And then the

story becomes more complicated. You can't really make the truncations that would give you the Boltzmann equation. You need to go by some other set of approximations, such as the Vlasov equation, that you will see in the problem set. And the actual theory that one can write down for liquids is very complicated. So it's - yeah.

AUDIENCE: Can it get more complicated yet if, based on the same framework, isn't this similar to, like, the formulation of the neutron transport equation, where if you have, in addition to scattering, you have particles that don't attract, but you have fission events and you have inelastic center?

PROFESSOR: OK. Yes. That's good.

AUDIENCE: And then your H would change dramatically, right?

PROFESSOR: Yes. That's right. Yes.

AUDIENCE: But my point is that it's the same framework.

PROFESSOR: Yes. Yes. So basically, you're right. One assumption that I have here is that the number of particles is conserved. We say that if the particles can break or diffuse, one has to generalize this approximation. And there would be an appropriate generalization of the Boltzmann equation to cover those terms. And I don't have a problem set related to that, but maybe you are suggesting that I should provide one.

[LAUGHTER]

OK. That's a good point. OK? All right. So we have made some assumptions here. And I guess my claim is that this equation-- well, what equation? So basically, what I'm going to do, I call the right-hand side of this equation after this assumption of molecular chaos, CFF. For simplicity, let me call this bunch of first derivatives that act on f_1 L for Liouvillian. It's kind of like a Liouville operator.

So I have an equation that is the bunch of first derivatives acting on f_1 , which is essentially a linear partial differential equation in six or seven dimensions,

depending on how you count time-- is equal to a non-linear integral on the right-hand side. OK? So this is the entity that is the Boltzmann equation.

Now, the statement is that this equation no longer has time reversal symmetry. And so if I were to solve this equation for the one function that I'm interested, after all, the f_1 , which really tells me how the particles stream from the left-hand side of the box to the right-hand side of the box, that equation is not reversible.

That is, the solution to this equation will indeed take the density that was on the left-hand side of the box and uniformly distribute it on the two sides of the box. And that will stay forever. That's, of course, a consequence of the various approximations that we made here, removing the reversibility. OK?

And the statement to show that mathematically is that if there is a function-- and now, for simplicity, since from now on, we don't care about f_2 , f_3 , et cetera, we are only caring about f_1 , I'm going to drop the index one and say that if there is a density f , which is really my f_1 , that satisfies the Boltzmann-- that is, I have a form such as this-- then there is a quantity H that only depends on time that, under the evolution that is implicit in this Boltzmann equation will always decrease, where H is an integral over the coordinates and momenta that are in this function f of p , q , and t .

And all I need to do is to multiply by the log. And then all of these are vectors if I drop it for saving time. And we said that once we recognize that f , up to a normalization, is the same thing as the one-particle probability-- and having previously seen objects such as this in probability theory corresponding to entropy information, entropy of mixing, you will not be surprised at how this expression came about.

And I had told you that in each one of the four sections, we will introduce some quantity that plays the role of entropy. So this is the one that is appropriate to our section on kinetic theory. So let's see how we go about--

AUDIENCE: Question.

PROFESSOR: Yes?

AUDIENCE: When you introduced the-- initialized the concept of entropy of information of the message that's $p \log p$, [INAUDIBLE] entropy. And as far as I understand, this concept came up in, like, mid 20th century at least, right?

PROFESSOR: I think slightly later than that. Of that order, yes.

AUDIENCE: OK. And these are Boltzmann equations?

PROFESSOR: Yes.

AUDIENCE: And so this was, like, 100 years before, at least?

PROFESSOR: Not 100 years, but maybe 60 years away, yeah. 50, 60.

AUDIENCE: OK. I'm just interested, what is the motivation for picking this form of functionality, that it's integral of $f \log f$ rather than integral of something else?

PROFESSOR: OK. I mean, there is another place that you've seen it, that has nothing to do with information. And that's mixing entropy. If I have $[n_1, n_2]$ one particles of one type and two particles of another type, et cetera, and mixed them up, then the mixing entropy, you can relate to a form such as this. It is some $n_i \log n_i$. OK? Fine.

It's just for discrete variables, it makes much more sense than for the continuum. But OK. So let's see if this is indeed the case. So we say that dH by dt is the-- I have to take the time derivative inside. And as we discussed before, the full derivative becomes a partial derivative inside the integral. And I have either acting on f and acting on $\log f$, which will give me 1 over f , canceling the f outside. And this term corresponded to the derivative of the normalization, which is fixed.

And then we had to replace what we have for df by dt . And for df by dt , we either have the Poisson bracket of the one-particle Hamiltonian with f coming from the L part of the equation, once we get d by dt on one side. And then the collision part that depends on two f 's. And we have to multiply this by f .

And again, let's remind you that f really has argument-- oops, $\log f$ -- that has argument p_1 . I won't write q_1 and t .

And the next thing that we did at the end of last lecture is that typically, when you have integral of a Poisson bracket by doing some combination of integrations by part, you can even eventually show that this contribution is 0. And really, the important thing is the integration against the collision term of the Boltzmann equation.

So writing that explicitly, what do we have? We have that it is integral d^3 . Let's say q_1 . Again, I don't really need to keep track of the index on this dummy variable. I have the integral over q_1 . I have the two integral over p_1 . Let's keep it so that it doesn't confuse anybody.

Then I have the collision term. Now, the collision term is a bunch of integrals itself. It involves integrals over p_2 , integral over an impact vector b , the relative velocity. And then I had minus f of p_1 , f of p_2 for the loss in the channel because of the collisions, plus the addition to the channel from the inverse collisions. And this whole thing has to be multiplied by $\log f$ evaluated at p_1 . OK?

So that's really the quantity that we are interested. And the quantity that we are interested, we notice, has some kind of a symmetry with respect to indices one and two. Something to do with prime and un-prime coordinates. But multiplied this bracket that has all of these symmetries, with this function that is only evaluated for one of the four coordinates for momenta that are appearing here.

So our next set of tasks is to somehow make this as symmetric as possible. So the first thing that we do is we exchange indices one and two, because really, these are simply dummy integration variables. I can call the thing that I was calling p_1 p_2 , and vice versa. And the integration then becomes the integral over q .

Maybe that's why it's useful not to have, for the q , any index, because I don't care to change the index for that. $d^3 p_2, p_1$. Doesn't really matter in what order I write them. $d^2 b$ is a relative separation. Again, it's an impact parameter. I can call this v_1

minus v_2 . But since it's an absolute value, it doesn't matter really.

And so the only thing that happens is that I will have-- essentially, the product is symmetric. The only thing that was separately f of p_1 before now becomes f of p_2 .

And then what I can do is I can say, OK, those are really two forms of the same integral, and I can average them, and write the average, which is $1/2$ the integral $d^3 q$, $d^3 p_1$, $d^3 p_2$, $d^2 b$, relative velocity, minus f of p_1 , f of p_2 , f of p_1 prime, f of p_2 prime. And then I have a log of f of p_1 plus log of f of p_2 . And I have divided through $1/2$. OK?

So what I will do is to now write the next line. And then we'll spend some time discussing what happens. So artificially, what will happen looks like I essentially move prime and un-prime things the same way that I exchanged one and two indices before. I will exchange the superscripts that are prime or non-prime. And if I were to do that, I will get an integral $d^3 q$, d^3 , let's say, p_1 prime, $d^3 p_2$ prime, $d^2 b$, v_2 minus v_1 prime, minus f of p_1 -- [SIGHS]

--prime, f of p_2 prime, plus f of p_1 , f of p_2 . And then here, we would have log of f of p_1 prime, log of f of p_2 prime. OK?

So this is a statement that just saying, I moved the primes and un-primes, it's very easy to come up with this answer. What you have to think very much about what that means. Essentially, what I had originally was integration invariables p_1 and p_2 . And p_1 prime and p_2 prime were never integration variables. So there was not symmetry at the level that I have written between one and two, which are integration variables, and one prime and two prime, which are the quantities that, through the collision operator, are related to p_1 and p_2 .

So essentially, p_1 prime and p_2 prime are the kind of functions that we have written here. Here, I had p_1 double prime, p_2 double prime. But up to sort of inverting some signs, et cetera, this is the kind of functional relationship that you have between p primes and p 's. OK?

So now we have to worry about a number of things. The simplest one is actually what happened to this factor here-- the relative velocity. Now think back about collisions that you have in the center of mass. In the center of mass, two particles come with velocities that are opposite each other. They bang onto each other, go different direction. The magnitude of the velocities does not change an inelastic collision, which is what we are looking at. Right? So after the collision, what we have is that $p_2' - p_1'$ is the same thing as p_1 and minus p_2 for elastic conditions.

So as far as this factor is concerned, I could very well replace that with v_1 minus v_2 without the primes. It really doesn't make any difference.

Another thing that you can convince yourself is that from the perspective of the picture that I drew, there was some kind of an impact vector B . You say, well, what happens if I look at from the perspective of the products of the interaction? So the products of the interaction are going in a different direction. I would have to redraw my coordinates that would correspond to A' and B' .

But what you can convince yourself that essentially, the impact parameter, as far as these products is concerned, is just a rotated version of the original one without changing its magnitude. So you have also that magnitude of b' is the same thing as magnitude of b . There's a rotation, but the impact parameter does not change. If you like, that is also related to conservation of energy, because if it wasn't so, the potential energy would be different right after the collisions. OK?

So I really was kind of careless here. I should have written $d^2 b'$. But I could have written also $d^2 b$. It didn't make any difference. It's the same.

And then, of course, these functions-- essentially, you can see that you say, OK, I start with, here, p_1 and p_2 . And these were functions of p_1 and p_2 . And here, I have reversed this. p_1' and p_2' are the integration variables. p_1 and p_2 are functions of p_1' and p_2' that, in principle, are obtained by inverting this set of equations. Right?

But it really, again, doesn't matter, because once you have specified what incoming p_1 , p_2 , and b are, you know what the outcome of the collision is. So it's really the same function. Whether you are inverting it, you will, again, presume the different location of the arguments, but you will have the same function that would tell you what the inverse relationship is. So really, these functional forms are the same as the functional forms that we had before.

And finally, I claim that in order to go through this change of variable, I have to also deal with the Jacobian of the transformation. But that we have the $d^3p_1 d^3p_2$ is the same thing as $d^3p_1' d^3p_2'$. And that, in some sense, you can sort of also think about what we were doing before with the Liouville equation maintaining the volume of phase space. So I have some volume of phase space prior to the collision. After the collision, I would need to have the same volume.

There is, of course, in the volumes that we are considering for the case of the Liouville operator, in addition, products of dq_1 , dq_2 . But you can see that all of those things are really put into the same volume when we are considering the collision. So really, the only thing that is left is that the Liouvilian preservation of the volume of phase space would say that the Jacobian here, you can also ignore. Yes.

AUDIENCE: So are you assuming that this transformation [INAUDIBLE] is [? an article, ?] because it's [INAUDIBLE]?

PROFESSOR: Yes. So everything that we do is within the framework of a Hamiltonian evolution equation. OK? So if you like, these kinds of things that I'm saying here sometimes are said to be consequence of microscopic reversibility, in that I'm sort of using, in this panel over here, things that are completely Newtonian and microscopic. The place that I use statistical argument was over here. OK?

So essentially, having done that, you can basically also rename variables. Things that I was calling p_1' I can call p_1 . It's another renaming of some variable. And again, it really doesn't matter which relative velocity I write down. They're all the same thing.

So what will happen is that I will get minus f of p_1 , f of p_2 , plus f of p_1 prime, f of p_2 prime. And the only thing that happened is that the logs-- oops. I should have maintained. I don't know why-- where I made a mistake. But I should ultimately have come up with this answer.

So what I wanted to do, and I really have to maybe look at it again, is do one more step of symmetrization. The answer will come down to an integral over p_1 and p_2 , d^2b , at relative velocity. We would have minus f of p_1 , f of p_2 , plus f of p_1 prime, f of p_2 prime.

So this is our collision operator that we have discussed here-- the difference between what comes in and what comes out. And from the first part here, we should get log of f of p_1 , plus log of f of p_2 , which is the same thing as the log of the product. And if I had not made errors, the subtraction would have involved f of p_1 prime, f of p_2 prime.

And the idea is that if you think about the function log of something as a function of that something, the log is a monotonic function. So that if you were to evaluate the log at two different points, the sign of the log would grow in the same way as the sign of the two points, which means that here, you have the difference of two logs.

This is like log of s_1 minus log of s_2 . And it is multiplied with minus s_1 minus s_2 . So if s_1 is greater than s_2 , log s_1 would be greater than log s_2 . If s_1 is less than s_2 , log s_1 would be less than log s_2 . In either case, what you would find is that the product of the two brackets is negative. And this whole thing has to be negative. OK.

So the thing that confused me-- and maybe we can go back and look at is that probably, what I was doing here was a little bit too impatient, because when I changed variables, p_1 prime f of p_2 . I removed the primes. Log. Reduced primes. Actually, I had it here. I don't know why I was worried. Yeah. So the signs are, I think, right, that when I add them, I would get the subtraction.

So in one case, I had the f 's of the p 's with the negative sign. After I did this change, they appear with the positive sign. So there is the sign difference between the two of

them. OK?

So we have a function, therefore, that, just like entropy throughout the process, will increase. This is like a negative entropy. If I were to really solve this set of equations, starting with the f that describes things that are distributed in this box, and then allow it to expand into the other box, it will follow-- this solution will follow some particular trajectory that will ultimately no longer change as a function of time. It will not go back and forth. It will not have the full reversibility that these set of equations does have.

Now, that's actually not a bad thing. It is true, indeed, that these equations are reversible. And there is a theorem that if you wait sufficiently long, this will go back into here. But the time it take for that to happen grows something of the order of the size of the system divided these. The time, up to various pre-factors, would grow exponentially with the number of particles that you have. And remember that you have of the order of 10^{23} particles.

So there is, indeed, mathematically rigorously, a recursion time if you really had this box sitting in vacuum and there was no influence on the walls of the container. After many, many, many, many ages of the universe, the gas would go back for an instant of time in the other box, and then would go over there. But that's something that really is of no practical relevance when we are thinking about the gas and how it expands. OK?

But now let's see, with these approximations, can we do better. Can we figure out exactly how long does it take for the gas to go from one side to the other side. And what is the shape of the streamlines and everything else as this process is taking place? Yes.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yes. Yes. So if you like, we want to think of this in terms of information, in terms of entropy. It doesn't matter. It is something that is changing as a function of time. And the timescale by which that happens has to do with the timescales that we set up

and imprinted into this equation. OK?

So the next step that I have to answer is, what is that time step? And in particular, if I look at this equation by itself, I would say that it has two timescales. On the left-hand side, you have this τ that has to do with the size of the box. We said it's very long.

On the right-hand side, it has the time that I have to wait for a collision to occur, which is certainly much shorter than this time, for an ordinary gas. And so we expect, therefore, that the right-hand side of this equation would determine things much more rapidly compared to the left-hand side of the equation. And we want to now quantify that and make that into something that is relevant to things that we know and understand about a gas expand. OK?

So let's sort of think about, well, what would happen? We said that this H , as a function of time, is going in one direction. Presumably, after some time-- we don't know how long here, and that's a good question-- eventually, hopefully, it will reach an equilibrium, like, the entropy will cease to change. And the question is, what is this equilibrium?

So let's see if we can gain some information about equilibrium. So we said that dH by dt presumably has to be 0, because if it is not 0, then it can continue to decrease. You are not quite where you want to be.

Now let's look at what is the condition for dH by dt to be 0. H is this integral that I have to perform over the entire phase space. Now, each element of that integral evaluated at some particular location in phase space is by itself positive. This argument that we had before is not about the entire integral. It's valid about every little piece of the integration that I am making. It has a particular sign.

So if the entire integral has to be 0, every little piece of it has to be individually 0. So that means that I would require that \log of f at any q evaluated, let's say, for p_1 and p_2 should be the same thing as \log of f evaluated at p_1 prime and p_2 . This would be true for all q . Right?

You look at this and you say, well, what? This seems kind of-- yes?

AUDIENCE: What about the other one in parentheses, beside the integral?

PROFESSOR: It is the same thing. So this is s_1 minus s_2 . And the other is log of s_1 minus s_2 . So I'm saying that the only time it is 0 is s_1 is equal to s_2 . When s_1 equals to s_2 , both parentheses are 0. Although, even if it wasn't, it would have been sufficient for one parentheses to be 0. But this is necessary, because the sign is determined by this. OK?

Now, you look at this and you say, well, what the he-- what does this mean? How can I-- and the functions p_1 prime and p_2 prime I have to get by integrating Newton's equations. I have no idea how complicated they are. And you want me to solve this equation. Well, I tell you that the answer to this is actually very simple.

So let me write the answer to it in stages. Log of f of p and q . So it's a function of p and q that I'm after that have to be evaluated at p_1 , p_2 , and then should be equal to p_1 prime and p_2 prime for any combination of p_1 , p_2 , et cetera, that I choose.

Well, I say, OK, I will put any function of q that I like. And that will work, because it's independent of p . So I have minus α , minus α is the same as minus α , minus α . OK? Then I claim that, OK, I will put a minus γ dot p . γ could be dependent on q . Dot p .

And I claim that that would work, because that term would give me γ dot p_1 plus p_2 , γ dot p_1 prime plus p_2 prime. What do I know for sure? Momentum is conserved. p_1 plus p_2 is the same thing as p_1 prime plus p_2 prime. It's just the conservation of momentum. I know that to be true for sure.

And finally, kinetic energy is conserved. Once I'm away from the location where the interaction is taking place, I can choose any function of q multiplied with p squared over $2m$. And then on the left-hand side, I would have the sum of the incoming kinetic energies.

On the right-hand side, I would have the sum of outgoing kinetic energies. So what

have I done? I have identified quantities that are conserved in the collision. So basically, I can keep going if I have additional such things. And so basically, the key to this is to identify collision-conserved quantities.

And so momentum, energy, the first one corresponds to the number of particles. These are quantities that are conserved. And so you know that this form will work, which means that I have a candidate for equilibrium, which is that f of p and q should have a form that is exponential of some function of q that I don't-- at this stage, could be quite arbitrary. And some other function of q times p squared minus $2m$. And let me absorb this other gamma term. I can certainly do so by putting something like π of 1 squared over $2m$. OK?

And this certainly is a form that will set the right-hand side of the H theorem to 0. And furthermore, it sets the right-hand side of the Boltzmann equation to 0, because, again, for the Boltzmann equation, all I needed was that the product of two f 's should be the same before and after the collision. And this is just the logs. If I were to exponentiate that, I have exactly what I need.

So basically, this also sets the right-hand side of the Boltzmann equation to 0. And this is what I will call a local equilibrium. What do I mean by that? Essentially, we can see that right now, I have no knowledge about the relationship between different points in space, because these α , β , and π are completely arbitrary, as far as my argumentation so far is concerned. So locally, at each point in q , I can have a form such as this. And this form, you should remind you it's something like a Boltzmann weight of the kinetic energy, but moving in some particular direction.

And essentially, what this captures is that through relaxing the right-hand side of the Boltzmann equation, we randomize the magnitude of the momenta, so that the magnitudes of the kinetic energy are kind of Boltzmann-distributed. This is what this term does. If I say that this term being much larger than this term is the more important one. If I need neglect this, then this is what happens.

I will very rapidly reach a situation in which the momenta have been randomized through the collisions. Essentially, the collisions, they change the direction of the

momenta. They preserve the average of the momenta. So there is some sense of the momentum of the incoming thing that is left over still. But there is some relaxation that took place.

But clearly, this is not the end of the story, because this f that I have written here-- let's say local equilibrium. f local equilibrium does not satisfy the Boltzmann equation. Right? It set the right-hand side of the Boltzmann equation to 0. But it doesn't set the left-hand side. The left-hand side completely is unhappy with this.

For the left-hand side to be 0, I would require the Poisson bracket of H_1 and f to be 0. After all, the Liouvillian operator had the Poisson bracket of H_1 and f . And we have seen functions that satisfy this have to be of the form f_1 of H . And in our case, my f_1 is simply the kinetic energy of one particle, plus the potential due to the box. OK?

So you can see that the only way that I can make this marriage with this is to have a global equilibrium, where f for the global equilibrium is proportional, let's say, to exponential of minus beta p squared over $2m$ plus U of q . OK?

So I identify a piece that looks like this p squared over $2m$. I have to make π to be 0 in order to achieve this. I have to make beta to be independent of q . I have to make alpha of f marriage with beta to give me the potential. OK?

So essentially, what happens is this. I open this hole. The gas starts to stream out. And then the particles will collide with each other. The collision of particles over some timescale that is related to this collision time will randomize their momenta so that locally, suddenly, I have a solution such as this.

But then the left-hand side of the Boltzmann equation takes over. And over a timescale that is much longer, it will then change the parameters of this. So basically, these parameters should really be thought of as functions of time, whose evolution changes. So initially, maybe locally over here, there is a stream velocity that goes towards the right. So I should have an average at that time that prefers to go to the right.

I wait sufficiently long. Presumably, the whole thing comes to equilibrium. And that average goes to 0 [INAUDIBLE]. So presumably, the next thing that I need to do in order to answer the question, how does this thing come to equilibrium, is to find out how the left-hand side of the equation manipulates these parameters alpha, beta, and pi. But that's not quite consistent either.

Because I first assume that the right-hand side is 0, and then said, oh, that's not a good solution. Let's go and see what the left-hand side does. But then if the left-hand side is changing things, I can't be over here where dH by dt is 0. So I have not done things consistently yet, although I have clearly captured the whole lot of what I want to describe eventually. OK?

So let's see how we can do it systematically.

AUDIENCE: Question.

PROFESSOR: Yes.

AUDIENCE: When you [? write ?] the condition for [INAUDIBLE], [? did you ?] write f_1 of Hamiltonian. And ends up Hamiltonian [INAUDIBLE] includes interaction terms.

PROFESSOR: It's the one-particle Hamiltonian.

AUDIENCE: Oh.

PROFESSOR: Right. So by the time I got to the equation for f_1 , I have H_1 . H_1 does not have a second particle in it to collide with. OK?

AUDIENCE: Then equilibrium system's distribution of particles in container does not depend on their interactions after the [INAUDIBLE]?

PROFESSOR: In the story that we are following here, yes, it does not. So this is an approximation that actually goes with the approximations that we made in neglecting some of the higher order terms, with nd cubed being much less than 1, and not looking at things at resolution that is of the order of d . And we would imagine that it is only at the resolutions--

AUDIENCE: [INAUDIBLE] neglected [INAUDIBLE] atomized gas conjoined onto the molecules. Something like that.

PROFESSOR: That would require a different description.

AUDIENCE: Yeah. It would require [INAUDIBLE].

PROFESSOR: Yeah. So indeed. Eventually, in the description that we are going to get, we are looking at the distribution that would be appropriate to a dilute gas and which interactions are not seen. OK? All right.

So let's see how we can do it better. So clearly, the key to the process is that collisions that take place frequently at timescale τ randomize a lot of things. But they cannot randomize quantities that are conserved in collisions. So those are the quantities that we have to look at.

And again, these are quantities that are conserved in collision, as opposed to quantities that are conserved according to H1. Like, momentum is conserved in the collision. Clearly, momentum was not conserved in H1 because of the size of the box and did not appear in the eventual solution. OK. Let's see what we get.

So we are going to make a couple of definitions. First, we define a local density. I say that I take my solution to Boltzmann equation. And now I don't know what it is. But whatever it is, I take the solution to the Boltzmann equation as a function of p and q and t . But all I am interested is, at this particular location, have the particles arrived yet, what's the density. So to get that, I essentially integrate over the momenta, which is the quantity that I am not interested. OK? Obvious.

I define-- maybe I need the average kinetic energy at that location. So maybe, in general, I need the average or some other quantity that I define at location q and t . What I do is I integrate at that position f of p , and q , and t , and this function O . Actually, I also define a normalization. I divide by n of q and t . OK?

Now, I implicitly used something over there that, again, I define here, which is a collision-conserved quantity, is some function of momentum-- I could put q also--

such that when evaluated for p_1 plus p_2 for any pair of p_1 's and p_2 's that I take is the same thing as when evaluated for functions of p_1 and p_2 that would correspond to the outcomes of the collision from or giving rights to p_1 , p_2 . OK?

And then there's a following theorem that we were going to use, which is that for any collision-conserved quantity, what I have is that I will define a quantity J , which is a function of q and t in principle, which is obtained by integrating over all momenta. χ -- let's say, evaluated at p_1 ; it could be evaluated at q -- times this collision operator-- C of f , f .

And the answer is that for any collision-conserved quantity, if you were to evaluate this, the answer would be 0. OK? So let's write that down. So J of q and t is an integral over some momentum p_1 . Because of the collision integral, what do I have?

I have the integration over momentum. I have the integration over b . I have the relative velocity v_2 minus v_1 . And then I have the subtraction of interaction between p_1 and p_2 -- addition of the interactions between p_1 prime, p_2 prime. And this collision integral has to be multiplied with χ evaluated at p_1 . OK? Fine.

So then I will follow exactly the steps that we had in the proof of the H theorem. I have some function that only depends on one of four momenta that are appearing here. And I want to symmetrize it. So I change the index one to two. So this becomes $1/2$ of χ of p_1 plus χ of p_2 .

I do this transition between primed and un-primed. And when I do that, I pick this additional factor of minus χ of p_1 prime, χ of p_2 prime. And this becomes division by 2 times 2.

But then I stated that χ is a collision-conserved quantity, so that this term in the bracket is 0 for each point that I'm integrating. And hence, the answer is 0. Yes.

AUDIENCE: In the [INAUDIBLE] theorem, in the second [INAUDIBLE], do you want to put minus sign of the--?

PROFESSOR: There was a minus sign worry that I had somewhere.

AUDIENCE: I think in front of $1/2$, at the right term.

PROFESSOR: In front of the $1/2$, minus f of p_1 . You say I have to do this?

AUDIENCE: No, no, no. The next term.

PROFESSOR: Here?

AUDIENCE: Yeah. I mean, if you put minus sign in front of the $1/2$, then [INAUDIBLE]?

PROFESSOR: OK. Let's put a question mark here, because I don't want to go over that.

AUDIENCE: [INAUDIBLE] but you switched the order of the subtraction [? $1/2$. ?] So it's minus [INAUDIBLE].

PROFESSOR: But I changed also the arguments here.

AUDIENCE: Oh, OK.

PROFESSOR: Right? So there is some issue. But this eventual thing is correct. So we will go back to that. OK.

So what is the use of stating that for every conserved quantity, you have this? It will allow us to get equations that will determine how these parameters change as a function of time? OK. So let's see how that goes.

If then I have that Lf is C, f, f , and I showed that the integral of χ against C, f, f is 0, then I also know that the integral of χ against this L operator on f is 0. OK?

And let's remind you that the L operator is a bunch of first derivatives-- d by dt , d by dq , d by dp . And so I can use the following result. I can write the result as this derivative acting on both χ and f . And then I will get $\chi' f + f \chi'$. $\chi' f$ I have, so I have to subtract fL acting on χ . OK?

And proceeding a little bit. One more line maybe. $d^3 p$. I have dt plus-- well, actually, let me do the rest next time around.