8.334 Statistical Mechanics II: Statistical Physics of Fields
Spring 2008

Review Problems & Solutions

The test is 'closed book,' but if you wish you may bring a one-sided sheet of formulas. The intent of this sheet is as a reminder of important formulas and definitions, and not as a compact transcription of the answers provided here. If this privilege is abused, it will be revoked for future tests. The test will be composed entirely from a subset of the following problems. Thus if you are familiar and comfortable with these problems, there will be no surprises!

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1. *Scaling in fluids:* Near the liquid–gas critical point, the free energy is assumed to take the scaling form $F/N = t^{2-\alpha}g(\delta\rho/t^{\beta})$, where $t = |T - T_c|/T_c$ is the reduced temperature, and $\delta\rho = \rho - \rho_c$ measures deviations from the critical point density. The leading singular behavior of any thermodynamic parameter $Q(t, \delta\rho)$ is of the form t^x on approaching the critical point along the isochore $\rho = \rho_c$; or $\delta\rho^y$ for a path along the isotherm $T = T_c$. Find the exponents x and y for the following quantities:

• Any homogeneous thermodynamic quantity $Q(t, \delta\rho)$ can be written in the scaling form

$$Q(t, \delta\rho) = t^{x_Q} g_Q\left(\frac{\delta\rho}{t^{\beta}}\right).$$

Thus, the leading singular behavior of Q is of the form t^{x_Q} if $\delta\rho = 0$, i.e. along the critical isochore. In order for any Q to be independent of t along the critical isotherm as $t \to 0$, the scaling function for a large enough argument should be of the form

$$\lim_{x\to\infty} g_Q(x) = x^{x_Q/\beta},$$

so that

$$Q(0, \delta\rho) \propto (\delta\rho)^{y_Q}, \qquad \text{with} \qquad y_Q = \frac{x_Q}{\beta}.$$

(a) The internal energy per particle $\langle H\rangle/N$, and the entropy per particle $s = S/N$.

• Let us assume that the free energy per particle is

$$f = \frac{F}{N} = t^{2-\alpha} g\left(\frac{\delta\rho}{t^{\beta}}\right),$$

and that $T < T_c$, so that $\frac{\partial}{\partial T} = -\frac{1}{T_c}\frac{\partial}{\partial t}$. The entropy is then given by

$$s = -\left.\frac{\partial f}{\partial T}\right|_V = \frac{1}{T_c}\left.\frac{\partial f}{\partial t}\right|_{\rho} = \frac{t^{1-\alpha}}{T_c} g_S\left(\frac{\delta\rho}{t^{\beta}}\right),$$

so that $x_S = 1-\alpha$, and $y_S = (1-\alpha)/\beta$. For the internal energy, we have

$$f = \frac{\langle \mathcal{H} \rangle}{N} - Ts, \qquad \text{or} \qquad \frac{\langle \mathcal{H} \rangle}{N} \sim T_c\, s(1+t) \sim t^{1-\alpha} g_{\mathcal{H}}\left(\frac{\delta\rho}{t^\beta}\right),$$

therefore, $x_{\mathcal{H}} = 1-\alpha$ and $y_{\mathcal{H}} = (1-\alpha)/\beta$.

(b) The heat capacities $C_V = T\partial s/\partial T\,|_V$, and $C_P = T\partial s/\partial T\,|_P$.

- The heat capacity at constant volume

$$C_V = T\left.\frac{\partial S}{\partial T}\right|_V = -\left.\frac{\partial s}{\partial t}\right|_\rho = \frac{t^{-\alpha}}{T_c} g_{C_V}\left(\frac{\delta\rho}{t^\beta}\right),$$

so that $x_{C_V} = -\alpha$ and $y_{C_V} = -\alpha/\beta$.

To calculate the heat capacity at constant pressure, we need to determine first the relation $\delta\rho(t)$ at constant P. For that purpose we will use the thermodynamic identity

$$\left.\frac{\partial\delta\rho}{\partial t}\right|_P = -\frac{\left.\frac{\partial P}{\partial t}\right|_\rho}{\left.\frac{\partial P}{\partial\delta\rho}\right|_t}.$$

The pressure P is determined as

$$P = -\frac{\partial F}{\partial V} = \rho^2 \frac{\partial f}{\partial \delta\rho} \sim \rho_c^2\, t^{2-\alpha-\beta} g_P\left(\frac{\delta\rho}{t^\beta}\right),$$

which for $\delta\rho \ll t^\beta$ goes like

$$P \propto t^{2-\alpha-\beta}\left(1 + A\frac{\delta\rho}{t^\beta}\right), \quad \text{and consequently} \quad \left\{\begin{array}{l} \left.\dfrac{\partial P}{\partial t}\right|_\rho \propto t^{1-\alpha-\beta} \\ \left.\dfrac{\partial P}{\partial \delta\rho}\right|_t \propto t^{2-\alpha-2\beta} \end{array}\right. .$$

In the other extreme of $\delta\rho \gg t^\beta$,

$$P \propto \delta\rho^{(2-\alpha-\beta)/\beta}\left(1 + B\frac{t}{\delta\rho^{1/\beta}}\right), \quad \text{and} \quad \left\{\begin{array}{l} \left.\dfrac{\partial P}{\partial t}\right|_\rho \propto \delta\rho^{(1-\alpha-\beta)/\beta} \\ \left.\dfrac{\partial P}{\partial \delta\rho}\right|_t \propto \delta\rho^{(2-\alpha-2\beta)/\beta} \end{array}\right. ,$$

where we have again required that P does not depend on $\delta\rho$ when $\delta\rho \to 0$, and on t if $t \to 0$.

From the previous results, we can now determine

$$\left.\frac{\partial\delta\rho}{\partial t}\right|_P \propto \left\{\begin{array}{ll} t^{\beta-1} & \Longrightarrow \delta\rho \propto t^\beta \\ \delta\rho^{(\beta-1)/\beta} & \Longrightarrow t \propto \delta\rho^{1/\beta} \end{array}\right. .$$

From any of these relationships follows that $\delta\rho \propto t^{\beta}$, and consequently the entropy is $s \propto t^{1-\alpha}$. The heat capacity at constant pressure is then given by

$$C_P \propto t^{-\alpha}, \qquad \text{with} \qquad x_{C_P} = -\alpha \qquad \text{and} \qquad y_{C_P} = -\frac{\alpha}{\beta}.$$

(c) The isothermal compressibility $\kappa_T = \partial\rho/\partial P\mid_T/\rho$, and the thermal expansion coefficient $\alpha = \partial V/\partial T\mid_P/V$.

Check that your results for parts (b) and (c) are consistent with the thermodynamic identity $C_P - C_V = TV\alpha^2/\kappa_T$.

• The isothermal compressibility and the thermal expansion coefficient can be computed using some of the relations obtained previously

$$\kappa_T = \frac{1}{\rho}\left.\frac{\partial\rho}{\partial P}\right|_T = \frac{1}{\rho_c}\left.\frac{\partial P}{\partial\rho}\right|_T^{-1} = \frac{1}{\rho_c^3}t^{\alpha+2\beta-2}g_\kappa\left(\frac{\delta\rho}{t^\beta}\right),$$

with $x_\kappa = \alpha + 2\beta - 2$, and $y_\kappa = (\alpha + 2\beta - 2)/\beta$. And

$$\alpha = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_P = \frac{1}{\rho T_c}\left.\frac{\partial\rho}{\partial t}\right|_P \propto t^{\beta-1},$$

with $x_\alpha = \beta - 1$, and $y_\alpha = (\beta - 1)/\beta$. So clearly, these results are consistent with the thermodynamic identity,

$$(C_P - C_V)(t, 0) \propto t^{-\alpha}, \qquad \text{or} \qquad (C_P - C_V)(0, \delta\rho) \propto \delta\rho^{-\alpha/\beta},$$

and

$$\frac{\alpha^2}{\kappa_T}(t, 0) \propto t^{-\alpha}, \qquad \text{or} \qquad \frac{\alpha^2}{\kappa_T}(0, \delta\rho) \propto \delta\rho^{-\alpha/\beta}.$$

(d) Sketch the behavior of the latent heat per particle L, on the coexistence curve for $T < T_c$, and find its singularity as a function of t.

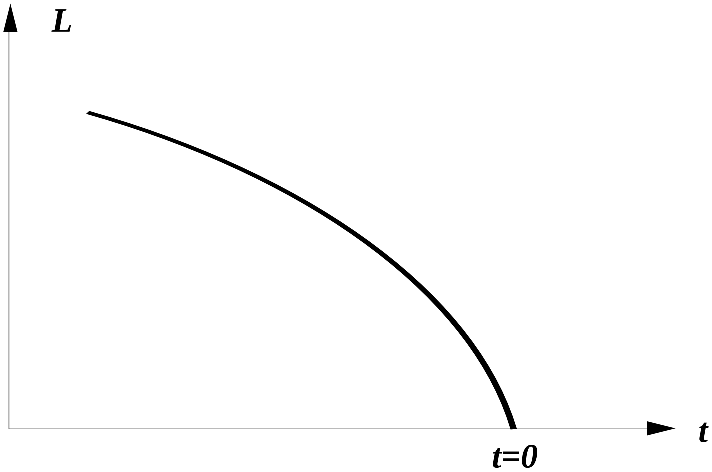
• The latent heat

$$L = T(s_+ - s_-)$$

is defined at the coexistence line, and as we have seen before

$$Ts_\pm = t^{1-\alpha}g_s\left(\frac{\delta\rho_\pm}{t^\beta}\right).$$

The density difference between the two coexisting phases is the order parameter, and vanishes as t^β, as do each of the two deviations $\delta\rho_+ = \rho_c - 1/v_+$ and $\delta\rho_- = \rho_c - 1/v_-$



of the gas and liquid densities from the critical critical value. (More precisely, as seen in (b), $\delta\rho|_{P=\text{constant}} \propto t^{\beta}$.) The argument of g in the above expression is thus evaluated at a finite value, and since the latent heat goes to zero on approaching the critical point, we get

$$L \propto t^{1-\alpha}, \qquad \text{with} \qquad x_L = 1 - \alpha.$$

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2. *The Ising model:* The differential recursion relations for temperature T, and magnetic field h, of the Ising model in $d = 1 + \epsilon$ dimensions are

$$\begin{cases} \dfrac{dT}{d\ell} = -\,\epsilon T + \dfrac{T^2}{2} & , \\ \dfrac{dh}{d\ell} = dh & . \end{cases}$$

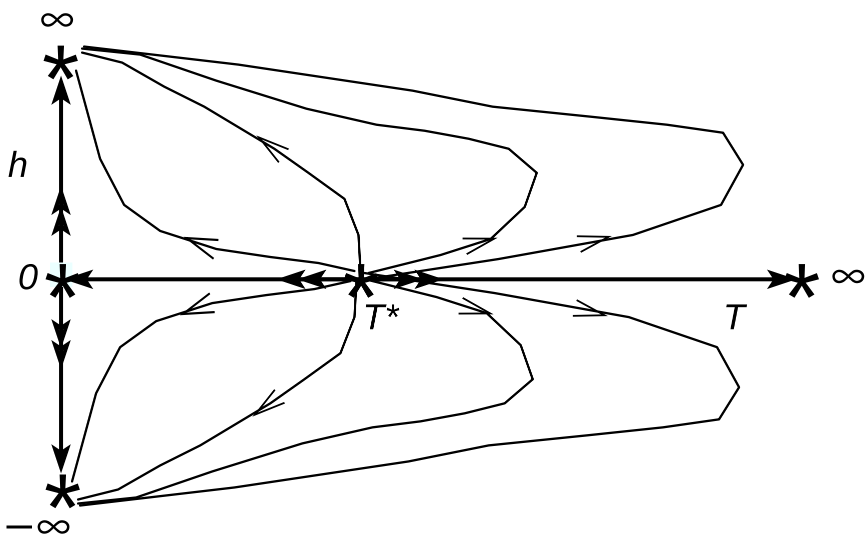
(a) Sketch the renormalization group flows in the (T, h) plane (for $\epsilon > 0$), marking the fixed points along the $h = 0$ axis.

• The fixed points of the flow occur along the $h = 0$ axis, which is mapped to itself under RG. On this axis, there are three fixed points: **(i)** $T^* = 0$, is the stable sink for the low temperature phase. **(ii)** $T^* \to \infty$, is the stable sink for the high temperature phase. **(iii)** There is a critical fixed point at $(T^* = 2\epsilon, h^* = 0)$, which is unstable. All fixed points are unstable in the field direction.

(b) Calculate the eigenvalues y_t and y_h, at the critical fixed point, to order of ϵ.

• Linearizing $T = T^* + \delta T$, around the critical fixed point yields

$$\begin{cases} \dfrac{d\delta T}{d\ell} = -\,\epsilon\,\delta T + T^*\delta T = \epsilon\delta T \\ \dfrac{dh}{d\ell} = (1+\epsilon)h \end{cases}, \quad \Longrightarrow \quad \begin{cases} y_t = +\,\epsilon \\ y_h = 1 + \epsilon \end{cases}.$$



(c) Starting from the relation governing the change of the correlation length ξ under renormalization, show that $\xi(t,h) = t^{-\nu} g_\xi\left(h/|t|^\Delta\right)$ (where $t = T/T_c - 1$), and find the exponents ν and Δ.

• Under rescaling by a factor of b, the correlation length is reduced by b, resulting in the homogeneity relation

$$\xi(t,h) = b\xi(b^{y_t}t, b^{y_h}h).$$

Upon selecting a rescaling factor such that $b^{y_t}t \sim 1$, we obtain

$$\xi(t,h) = t^{-\nu} g_\xi\left(h/|t|^\Delta\right),$$

with

$$\nu = \frac{1}{y_t} = \frac{1}{\epsilon}, \qquad \text{and} \qquad \Delta = \frac{y_h}{y_t} = \frac{1}{\epsilon} + 1.$$

(d) Use a hyperscaling relation to find the singular part of the free energy $f_{\text{sing.}}(t,h)$, and hence the heat capacity exponent α.

• According to hyperscaling

$$f_{\text{sing.}}(t,h) \propto \xi(t,h)^{-d} = t^{d/y_t} g_f\left(h/|t|^\Delta\right).$$

Taking two derivatives with respect to t leads to the heat capacity, whose singularity for $h = 0$ is described by the exponent

$$\alpha = 2 - d\nu = 2 - \frac{1+\epsilon}{\epsilon} = -\frac{1}{\epsilon} + 1.$$

(e) Find the exponents β and γ for the singular behaviors of the magnetization and susceptibility, respectively.

• The magnetization is obtained from the free energy by

$$m = -\left.\frac{\partial f}{\partial h}\right|_{h=0} \sim |t|^{\beta}, \quad \text{with} \quad \beta = \frac{d - y_h}{y_t} = 0.$$

(There will be corrections to β at higher orders in ϵ.) The susceptibility is obtained from a derivative of the magnetization, or

$$\chi = -\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim |t|^{-\gamma}, \quad \text{with} \quad \gamma = \frac{2y_h - d}{y_t} = \frac{1+\epsilon}{\epsilon} = \frac{1}{\epsilon} + 1.$$

(f) Starting the relation between susceptibility and correlations of local magnetizations, calculate the exponent η for the critical correlations ($\langle m(\mathbf{0})m(\mathbf{x})\rangle \sim |\mathbf{x}|^{-(d-2+\eta)}$).

• The magnetic susceptibility is related to the connected correlation function via

$$\chi = \int d^d\mathbf{x} \, \langle m(\mathbf{0})m(\mathbf{x})\rangle_c \, .$$

Close to criticality, the correlations decay as a power law $\langle m(\mathbf{0})m(\mathbf{x})\rangle \sim |\mathbf{x}|^{-(d-2+\eta)}$, which is cut off at the correlation length ξ, resulting in

$$\chi \sim \xi^{(2-\eta)} \sim |t|^{-(2-\eta)\nu}.$$

From the corresponding exponent identity, we find

$$\gamma = (2-\eta)\nu, \quad \Longrightarrow \quad \eta = 2 - y_t\gamma = 2 - 2y_h + d = 2 - d = 1 - \epsilon.$$

(g) How does the correlation length diverge as $T \to 0$ (along $h = 0$) for $d = 1$?

• For $d = 1$, the recursion relation for temperature can be rearranged and integrated, i.e.

$$\frac{1}{T^2}\frac{dT}{d\ell} = \frac{1}{2}, \quad \Longrightarrow \quad d\left(-\frac{2}{T}\right) = d\ell.$$

We can integrate the above expression from a low temperature with correlation length $\xi(T)$ to a high temperature where $1/T \approx 0$, and at which the correlation length is of the order of the lattice spacing, to get

$$-\frac{2}{T} = \ln\left(\frac{\xi}{a}\right) \quad \Longrightarrow \quad \xi(T) = a\exp\left(\frac{2}{T}\right).$$

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3. *Longitudinal susceptibility:* While there is no reason for the longitudinal susceptibility to diverge at the mean-field level, it in fact does so due to fluctuations in dimensions $d < 4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau–Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2\right] \qquad ,$$

describing an n–component magnetization vector $\vec{m}(\mathbf{x})$, in the ordered phase for $t < 0$.

(a) Let $\vec{m}(\mathbf{x}) = \left(\overline{m}+\phi_\ell(\mathbf{x})\right)\hat{e}_\ell+\vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

• With $\vec{m}(\mathbf{x}) = (\bar{m} + \phi_\ell(\mathbf{x}))\,\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\,\hat{e}_t$, and $\bar{m}$ the minimum of $\beta\mathcal{H}$,

$$\begin{aligned}\beta\mathcal{H} =& V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \int d^dx\Bigg\{\frac{K}{2}\left[(\nabla\phi_\ell)^2 + \left(\nabla\vec{\phi}_t\right)^2\right] + \left(\frac{t}{2} + 6u\bar{m}^2\right)\phi_\ell^2\\ &+\left(\frac{t}{2} + 2u\bar{m}^2\right)\vec{\phi}_t^{\;2} + 4u\bar{m}\left(\phi_\ell^3 + \phi_\ell\vec{\phi}_t^{\;2}\right) + u\left[\phi_\ell^4 + 2\phi_\ell^2\vec{\phi}_t^{\;2} + \left(\vec{\phi}_t^{\;2}\right)^2\right]\Bigg\}.\end{aligned}$$

Since $\bar{m}^2 = -t/4u$ in the ordered phase ($t < 0$), this expression can be simplified, upon dropping the constant term, as

$$\begin{aligned}\beta\mathcal{H} = \int d^dx\Bigg\{&\frac{K}{2}\left[(\nabla\phi_\ell)^2 + \left(\nabla\vec{\phi}_t\right)^2\right] - t\phi_\ell^2 + 4u\bar{m}\left(\phi_\ell^3 + \phi_\ell\vec{\phi}_t^{\;2}\right)\\ &+u\left[\phi_\ell^4 + 2\phi_\ell^2\vec{\phi}_t^{\;2} + \left(\vec{\phi}_t^{\;2}\right)^2\right]\Bigg\}.\end{aligned}$$

(b) Regard the quadratic terms in ϕ_ℓ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_ℓ and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U = 4u\overline{m}\int d^d\mathbf{x}\phi_\ell(\mathbf{x})\vec{\phi}_t(\mathbf{x})^2.$$

Write U in Fourier space in terms of $\phi_\ell(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

• We shall focus on the cubic term as a perturbation

$$U = 4u\bar{m}\int d^dx\phi_\ell(\mathbf{x})\,\vec{\phi}_t(\mathbf{x})^2\,,$$

which can be written in Fourier space as

$$U = 4u\bar{m} \int \frac{d^d q}{(2\pi)^d} \frac{d^d q'}{(2\pi)^d} \phi_\ell \left(-\mathbf{q} - \mathbf{q}'\right) \vec{\phi}_t \left(\mathbf{q}\right) \cdot \vec{\phi}_t \left(\mathbf{q}'\right).$$

(c) Calculate the Gaussian (bare) expectation values $\langle \phi_\ell(\mathbf{q}) \phi_\ell(\mathbf{q}') \rangle_0$ and $\langle \phi_{t,\alpha}(\mathbf{q}) \phi_{t,\beta}(\mathbf{q}') \rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

• From the quadratic part of the Hamiltonian,

$$\beta \mathcal{H}_0 = \int d^d x \frac{1}{2} \left\{ K \left[\left(\nabla \phi_\ell\right)^2 + \left(\nabla \vec{\phi}_t\right)^2 \right] - 2t\phi_\ell^2 \right\},$$

we read off the expectation values

$$\begin{cases} \left\langle \phi_\ell \left(\mathbf{q}\right) \phi_\ell \left(\mathbf{q}'\right) \right\rangle_0 = \dfrac{(2\pi)^d \delta^d \left(\mathbf{q} + \mathbf{q}'\right)}{Kq^2 - 2t} \\ \left\langle \phi_{t,\alpha} \left(\mathbf{q}\right) \phi_{t,\beta} \left(\mathbf{q}'\right) \right\rangle_0 = \dfrac{(2\pi)^d \delta^d \left(\mathbf{q} + \mathbf{q}'\right) \delta_{\alpha\beta}}{Kq^2} \end{cases},$$

and the corresponding susceptibilities

$$\begin{cases} \chi_\ell \left(\mathbf{q}\right)_0 = \dfrac{1}{Kq^2 - 2t} \\ \chi_t \left(\mathbf{q}\right)_0 = \dfrac{1}{Kq^2} \end{cases}.$$

(d) Calculate $\langle \vec{\phi}_t(\mathbf{q}_1) \cdot \vec{\phi}_t(\mathbf{q}_2) \ \ \vec{\phi}_t(\mathbf{q}'_1) \cdot \vec{\phi}_t(\mathbf{q}'_2) \rangle_0$ using Wick's theorem. (Don't forget that $\vec{\phi}_t$ is an $(n-1)$ component vector.)

• Using Wick's theorem,

$$\begin{aligned} &\left\langle \vec{\phi}_t \left(\mathbf{q}_1\right) \cdot \vec{\phi}_t \left(\mathbf{q}_2\right) \vec{\phi}_t \left(\mathbf{q}'_1\right) \cdot \vec{\phi}_t \left(\mathbf{q}'_2\right) \right\rangle_0 \equiv \left\langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\alpha} \left(\mathbf{q}_2\right) \phi_{t,\beta} \left(\mathbf{q}'_1\right) \phi_{t,\beta} \left(\mathbf{q}'_2\right) \right\rangle_0 \\ &= \left\langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\alpha} \left(\mathbf{q}_2\right) \right\rangle_0 \left\langle \phi_{t,\beta} \left(\mathbf{q}'_1\right) \phi_{t,\beta} \left(\mathbf{q}'_2\right) \right\rangle_0 + \left\langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\beta} \left(\mathbf{q}'_1\right) \right\rangle_0 \left\langle \phi_{t,\alpha} \left(\mathbf{q}_2\right) \phi_{t,\beta} \left(\mathbf{q}'_2\right) \right\rangle_0 \\ &+ \left\langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\beta} \left(\mathbf{q}'_2\right) \right\rangle_0 \left\langle \phi_{t,\alpha} \left(\mathbf{q}_2\right) \phi_{t,\beta} \left(\mathbf{q}'_1\right) \right\rangle_0 . \end{aligned}$$

Then, from part (c),

$$\begin{aligned} \left\langle \vec{\phi}_t \left(\mathbf{q}_1\right) \cdot \vec{\phi}_t \left(\mathbf{q}_2\right) \vec{\phi}_t \left(\mathbf{q}'_1\right) \cdot \vec{\phi}_t \left(\mathbf{q}'_2\right) \right\rangle_0 = \frac{(2\pi)^{2d}}{K^2} \Bigg\{ & (n-1)^2 \frac{\delta^d \left(\mathbf{q}_1 + \mathbf{q}_2\right) \delta^d \left(\mathbf{q}'_1 + \mathbf{q}'_2\right)}{q_1^2 q_1'^2} \\ + (n-1) \frac{\delta^d \left(\mathbf{q}_1 + \mathbf{q}'_1\right) \delta^d \left(\mathbf{q}_2 + \mathbf{q}'_2\right)}{q_1^2 q_2^2} &+ (n-1) \frac{\delta^d \left(\mathbf{q}_1 + \mathbf{q}'_2\right) \delta^d \left(\mathbf{q}'_1 + \mathbf{q}_2\right)}{q_1^2 q_2^2} \Bigg\}, \end{aligned}$$

since $\delta_{\alpha\alpha}\delta_{\beta\beta} = (n-1)^2$, and $\delta_{\alpha\beta}\delta_{\alpha\beta} = (n-1)$.

(e) Write down the expression for $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second-order in the perturbation U. Note that since U is odd in ϕ_ℓ, only two terms at the second order are non–zero.

• Including the perturbation U in the calculation of the correlation function, we have

$$\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle = \frac{\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\,e^{-U}\right\rangle_0}{\left\langle e^{-U}\right\rangle_0} = \frac{\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\left(1-U+U^2/2+\cdots\right)\right\rangle_0}{\langle(1-U+U^2/2+\cdots)\rangle_0}.$$

Since U is odd in ϕ_ℓ, $\langle U\rangle_0 = \langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\,U\rangle_0 = 0$. Thus, after expanding the denominator to second order,

$$\frac{1}{1+\langle U^2/2\rangle_0+\cdots} = 1-\left\langle\frac{U^2}{2}\right\rangle_0 + \mathcal{O}\left(U^3\right),$$

we obtain

$$\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle = \langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle_0 + \frac{1}{2}\left(\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\,U^2\right\rangle_0 - \langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle_0\left\langle U^2\right\rangle_0\right).$$

(f) Using the form of U in Fourier space, write the correction term as a product of two 4–point expectation values similar to those of part (d). Note that only connected terms for the longitudinal 4–point function should be included.

• Substituting for U its expression in terms of Fourier transforms from part (b), the fluctuation correction to the correlation function reads

$$\begin{aligned}
G_F(\mathbf{q},\mathbf{q}') &\equiv \langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle - \langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle_0 \\
&= \frac{1}{2}(4u\bar{m})^2\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\Big\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2) \\
&\times\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\Big\rangle_0 - \frac{1}{2}\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle_0\left\langle U^2\right\rangle_0,
\end{aligned}$$

i.e. $G_F(\mathbf{q},\mathbf{q}')$ is calculated as the connected part of

$$\begin{aligned}
\frac{1}{2}(4u\bar{m})^2\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\,&\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0 \\
&\times\left\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\right\rangle_0,
\end{aligned}$$

where we have used the fact that the unperturbed averages of products of longitudinal and transverse fields factorize. Hence

$$
\begin{aligned}
G_F(\mathbf{q},\mathbf{q}') = \frac{1}{2}(4u\bar{m})^2 \int & \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\left\langle \vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\right\rangle_0 \\
& \times\Big\{ \langle\phi_\ell(\mathbf{q})\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\rangle_0\,\langle\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0 \\
& + \langle\phi_\ell(\mathbf{q})\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0\,\langle\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\rangle_0\Big\} \\
= 2\times\frac{1}{2}(4u\bar{m})^2 \int & \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\left\langle \vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\right\rangle_0 \\
& \times \langle\phi_\ell(\mathbf{q})\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\rangle_0\,\langle\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\rangle_0 .
\end{aligned}
$$

Using the results of parts (c) and (d) for the two and four points correlation functions, and since $u^2\bar{m}^2 = -ut/4$, we obtain

$$
\begin{aligned}
G_F(\mathbf{q},\mathbf{q}') = 4u(-t)\int & \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\frac{(2\pi)^{2d}}{K^2}\Bigg\{(n-1)^2\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_2)\,\delta^d(\mathbf{q}_1'+\mathbf{q}_2')}{q_1^2 q_1'^2} \\
& + (n-1)\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_1')\,\delta^d(\mathbf{q}_2+\mathbf{q}_2') + \delta^d(\mathbf{q}_1+\mathbf{q}_2')\,\delta^d(\mathbf{q}_1'+\mathbf{q}_2)}{q_1^2q_2^2}\Bigg\} \\
& \times\frac{(2\pi)^d\,\delta^d(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)}{Kq^2-2t}\,\frac{(2\pi)^d\,\delta^d(\mathbf{q}'-\mathbf{q}_1'-\mathbf{q}_2')}{Kq^2-2t},
\end{aligned}
$$

which, after doing some of the integrals, reduces to

$$
\begin{aligned}
G_F(\mathbf{q},\mathbf{q}') = \frac{4u(-t)}{K^2}\Bigg\{ & (n-1)^2\frac{\delta^d(\mathbf{q})\,\delta^d(\mathbf{q}')}{4t^2}\left(\int\frac{d^dq_1}{q_1^2}\right)^2 \\
& + 2(n-1)\frac{\delta^d(\mathbf{q}+\mathbf{q}')}{(Kq^2-2t)^2}\int\frac{d^dq_1}{q_1^2(\mathbf{q}+\mathbf{q}_1)^2}\Bigg\}.
\end{aligned}
$$

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

• From the dependence of the first term (proportional to $\delta^d(\mathbf{q})\,\delta^d(\mathbf{q}')$), we deduce that this term is actually a correction to the unperturbed value of the magnetization, i.e.

$$
\overline{m} \to \overline{m}\left[1 - \frac{2(n-1)u}{Kt}\left(\int\frac{d^dq_1}{q_1^2}\right)\right],
$$

and does not contribute to the correlation function at non-zero separation. The spatially varying part of the connected correlation function is thus

$$
\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle = \frac{(2\pi)^d\,\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2-2t} + \frac{8u(-t)}{K^2}(n-1)\frac{\delta^d(\mathbf{q}+\mathbf{q}')}{(Kq^2-2t)^2}\int\frac{d^dq_1}{q_1^2(\mathbf{q}+\mathbf{q}_1)^2},
$$

leading to

$$\chi_\ell(\mathbf{q}) = \frac{1}{Kq^2 - 2t} + \frac{8u(-t)}{K^2}\frac{(n-1)}{(Kq^2-2t)^2}\int \frac{d^d q_1}{(2\pi)^d}\frac{1}{q_1^2(\mathbf{q}+\mathbf{q}_1)^2}.$$

(h) Show that for $d < 4$, the correction term diverges as q^{d-4} for $q \to 0$, implying an infinite longitudinal susceptibility.

• In $d > 4$, the above integral converges and is dominated by the large q cutoff Λ. In $d < 4$, on the other hand, the integral clearly diverges as $q \to 0$, and is thus dominated by small q_1 values. Changing the variable of integration to $\mathbf{q}_1' = \mathbf{q}_1/q$, the fluctuation correction to the susceptibility reads

$$\chi_\ell(\mathbf{q})_F \sim q^{d-4}\int_0^{\Lambda/q}\frac{d^d q_1'}{(2\pi)^d}\frac{1}{q_1'^2(\hat{\mathbf{q}}+\mathbf{q}_1')^2} = q^{d-4}\int_0^{\infty}\frac{d^d q_1'}{(2\pi)^d}\frac{1}{q_1'^2(\hat{\mathbf{q}}+\mathbf{q}_1')^2} + \mathcal{O}\left(q^0\right),$$

which diverges as q^{d-4} for $q \to 0$.

NOTE: For a translationally invariant system,

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle = \varphi(\mathbf{x}-\mathbf{x}'),$$

which implies

$$\begin{aligned}
\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle &= \int d^d x d^d x' e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle \\
&= \int d^d(x-x')\,d^d x' e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')+i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}'}\varphi(\mathbf{x}-\mathbf{x}') \\
&= (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\psi(\mathbf{q}).
\end{aligned}$$

Consider the Hamiltonian

$$-\beta\mathcal{H}' = -\beta\mathcal{H} + \int d^d x h(\mathbf{x})\phi(\mathbf{x}) = -\beta\mathcal{H} + \int\frac{d^d q}{(2\pi)^d}h(\mathbf{q})\phi(-\mathbf{q}),$$

where $-\beta\mathcal{H}$ is a translationally invariant functional of ϕ (a one-component field for simplicity), independent of $h(\mathbf{x})$. We have

$$m(\mathbf{x}=\mathbf{0}) = \langle\phi(\mathbf{0})\rangle = \int\frac{d^d q}{(2\pi)^d}\langle\phi(\mathbf{q})\rangle,$$

and, taking a derivative,

$$\frac{\partial m}{\partial h(\mathbf{q})} = \int\frac{d^d q'}{(2\pi)^d}\langle\phi(\mathbf{q}')\phi(\mathbf{q})\rangle.$$

At $h = 0$, the system is translationally invariant, and

$$\left.\frac{\partial m}{\partial h(\mathbf{q})}\right|_{h=0} = \psi(\mathbf{q}).$$

Also, for a uniform external magnetic field, the system is translationally invariant, and

$$-\beta\mathcal{H}' = -\beta\mathcal{H} + h\int d^dx\phi(\mathbf{x}) = -\beta\mathcal{H} + h\phi(\mathbf{q}=\mathbf{0}),$$

yielding

$$\chi = \frac{\partial m}{\partial h} = \int \frac{d^dq'}{(2\pi)^d}\left\langle \phi(\mathbf{q}')\,\phi(\mathbf{q}=\mathbf{0})\right\rangle = \psi(\mathbf{0}).$$

****

4. *Crystal anisotropy:* Consider a ferromagnet with a tetragonal crystal structure. Coupling of the spins to the underlying lattice may destroy their full rotational symmetry. The resulting anisotropies can be described by modifying the Landau–Ginzburg Hamiltonian to

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u\left(\vec{m}^2\right)^2 + \frac{r}{2}m_1^2 + v\,m_1^2\,\vec{m}^2\right],$$

where $\vec{m} \equiv (m_1, \cdots, m_n)$, and $\vec{m}^2 = \sum_{i=1}^n m_i^2$ ($d = n = 3$ for magnets in three dimensions). Here $u > 0$, and *to simplify calculations we shall set* $v = 0$ throughout.

(a) For a fixed magnitude $|\vec{m}|$; what directions in the n component magnetization space are selected for $r > 0$, and for $r < 0$?

• $r > 0$ discourages ordering along direction 1, and leads to order along the remaining $(n-1)$ directions.

$r < 0$ encourages ordering along direction 1.

(b) Using the saddle point approximation, calculate the free energies ($\ln Z$) for phases uniformly magnetized *parallel* and *perpendicular* to direction 1.

• In the saddle point approximation for $\vec{m}(\mathbf{x}) = m\hat{e}_1$, we have

$$\ln Z_{sp} = -V\min\left[\frac{t+r}{2}m^2 + um^4\right]_m,$$

where $V = \int d^d\mathbf{x}$, is the system volume. The minimum is obtained for

$$(t+r)\overline{m} + 4u\overline{m}^3 = 0, \qquad \Longrightarrow \qquad \overline{m} = \begin{cases} 0 & \text{for } t+r > 0 \\ \sqrt{-(t+r)/4u} & \text{for } t+r < 0 \end{cases}.$$

For $t + r < 0$, the free energy is given by

$$f_{sp} = -\frac{\ln Z_{sp}}{V} = -\frac{(t+r)^2}{16u}.$$

When the magnetization is perpendicular to direction 1, i.e. for $\vec{m}(\mathbf{x}) = m\hat{e}_i$ for $i \neq 1$, the corresponding expressions are

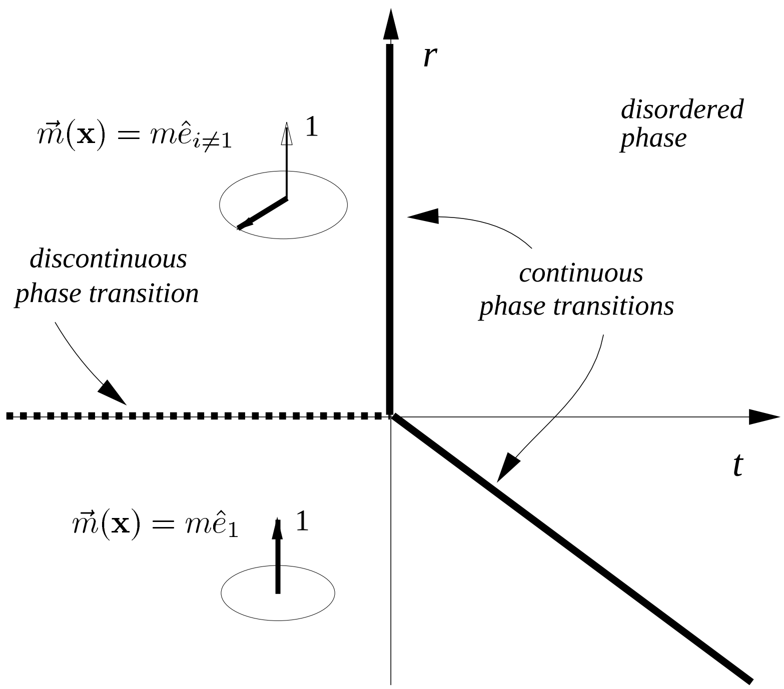
$$\ln Z_{sp} = -V \min \left[\frac{t}{2}m^2 + um^4\right]_m, \quad t\overline{m} + 4u\overline{m}^3 = 0, \quad \overline{m} = \begin{cases} 0 & \text{for } t > 0 \\ \sqrt{-t/4u} & \text{for } t < 0 \end{cases},$$

and the free energy for $t < 0$ is

$$f_{sp} = -\frac{t^2}{16u}.$$

(c) Sketch the phase diagram in the (t, r) plane, and indicate the phases (type of order), and the nature of the phase transitions (continuous or discontinuous).

• The saddle point phase diagram is sketched in the figure.



(d) Are there Goldstone modes in the ordered phases?

• There are no Goldstone modes in the phase with magnetization aligned along direction 1, as the broken symmetry in this case is discrete. However, there are $(n-2)$ Goldstone modes in the phase where magnetization is perpendicular to direction 1.

(e) For $u = 0$, and positive t and r, calculate the unperturbed averages $\langle m_1(\mathbf{q}) m_1(\mathbf{q}')\rangle_0$ and $\langle m_2(\mathbf{q}) m_2(\mathbf{q}')\rangle_0$, where $m_i(\mathbf{q})$ indicates the Fourier transform of $m_i(\mathbf{x})$.

• The Gaussian part of the Hamiltonian can be decomposed into Fourier modes as

$$\beta\mathcal{H}_0 = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \left[\frac{K}{2}q^2|\vec{m}(\mathbf{q})|^2 + \frac{t+r}{2}|m_1(\mathbf{q})|^2 + \sum_{i=2}^{n} \frac{t}{2}|m_i(\mathbf{q})|^2\right].$$

From this form we can easily read off the covariances

$$\begin{cases} \langle m_1(\mathbf{q}) m_1(\mathbf{q}') \rangle_0 = \dfrac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{t+r+Kq^2} \\ \langle m_2(\mathbf{q}) m_2(\mathbf{q}') \rangle_0 = \dfrac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{t+Kq^2} \end{cases}.$$

(f) Write the fourth order term $\mathcal{U} \equiv u \int d^d\mathbf{x} (\vec{m}^2)^2$, in terms of the Fourier modes $m_i(\mathbf{q})$.

• Substituting $m_i(\mathbf{x}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \exp(i\mathbf{q}\cdot\mathbf{x}) m_i(\mathbf{q})$ in the quartic term, and integrating over $\mathbf{x}$ yields

$$\mathcal{U} = u \int d^d\mathbf{x} \left(\vec{m}^2\right)^2 = u \int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}} \sum_{i,j=1}^{n} m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3).$$

(g) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_1(\mathbf{q}) m_1(\mathbf{q}') \rangle$. (You can leave your answers in the form of some integrals.)

• In first order perturbation theory $\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_0 - (\langle \mathcal{O}\mathcal{U} \rangle_0 - \langle \mathcal{O} \rangle_0 \langle \mathcal{U} \rangle_0)$, and hence

$$\begin{aligned} \langle m_1(\mathbf{q}) m_1(\mathbf{q}') \rangle = {} & \langle m_1(\mathbf{q}) m_1(\mathbf{q}') \rangle_0 - u \int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}} \sum_{i,j=1}^{n} \\ & \left(\langle m_1(\mathbf{q}) m_1(\mathbf{q}') m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \rangle_0^c \right) \\ = & \frac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{t+r+Kq^2} \left\{ 1 - \frac{u}{t+r+Kq^2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left[\frac{4(n-1)}{t+Kk^2} + \frac{4}{t+r+Kk^2} + \frac{8}{t+r+Kk^2} \right] \right\} \end{aligned}$$

The last result is obtained by listing all possible contractions, and keeping track of how many involve m_1 versus $m_{i\neq 1}$. The final result can be simplified to

$$\langle m_1(\mathbf{q}) m_1(\mathbf{q}') \rangle = \frac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{t+r+Kq^2} \left\{ 1 - \frac{u}{t+r+Kq^2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left[\frac{n-1}{t+Kk^2} + \frac{3}{t+r+Kk^2} \right] \right\}$$

(h) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_2(\mathbf{q}) m_2(\mathbf{q}') \rangle$.

• Similar analysis yields

$$\begin{aligned} \langle m_2(\mathbf{q}) m_2(\mathbf{q}') \rangle = {} & \langle m_2(\mathbf{q}) m_2(\mathbf{q}') \rangle_0 - u \int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}} \sum_{i,j=1}^{n} \\ & \left(\langle m_2(\mathbf{q}) m_2(\mathbf{q}') m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \rangle_0^c \right) \\ = & \frac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{t+Kq^2} \left\{ 1 - \frac{u}{t+Kq^2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left[\frac{4(n-1)}{t+Kk^2} + \frac{4}{t+r+Kk^2} + \frac{8}{t+Kk^2} \right] \right\} \\ = & \frac{(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')}{t+Kq^2} \left\{ 1 - \frac{u}{t+Kq^2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left[\frac{n+1}{t+Kk^2} + \frac{1}{t+r+Kk^2} \right] \right\}. \end{aligned}$$

(i) Using the above answer, identify the inverse susceptibility χ_{22}^{-1}, and then find the transition point, t_c, from its vanishing to first order in u.

• Using the fluctuation–response relation, the susceptibility is given by

$$\begin{aligned}\chi_{22} &= \int d^d\mathbf{x}\,\langle m_2(\mathbf{x})m_2(\mathbf{0})\rangle = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\,\langle m_2(\mathbf{q})m_2(\mathbf{q}=0)\rangle \\ &= \frac{1}{t}\left\{1 - \frac{u}{t}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n+1}{t+Kk^2} + \frac{1}{t+r+Kk^2}\right]\right\}.\end{aligned}$$

Inverting the correction term gives

$$\chi_{22}^{-1} = t + 4u\int \frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n+1}{t+Kk^2} + \frac{1}{t+r+Kk^2}\right] + O(u^2).$$

The susceptibility diverges at

$$t_c = -4u\int \frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n+1}{Kk^2} + \frac{1}{r+Kk^2}\right] + O(u^2).$$

(j) Is the critical behavior different from the isotropic $O(n)$ model in $d < 4$? In RG language, is the parameter r *relevant* at the $\mathcal{O}(n)$ fixed point? In either case indicate the universality classes expected for the transitions.

• The parameter r changes the symmetry of the ordered state, and hence the universality class of the disordering transition. As indicated in the figure, the transition belongs to the $O(n-1)$ universality class for $r > 0$, and to the Ising class for $r < 0$. Any RG transformation must thus find r to be a relevant perturbation to the $O(n)$ fixed point.

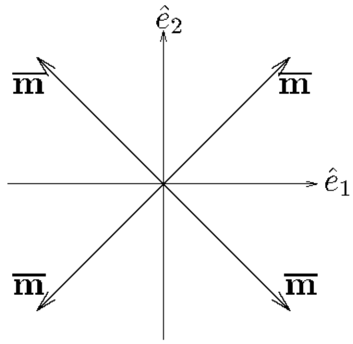
****

5. *Cubic anisotropy– Mean-field treatment:* Consider the modified Landau–Ginzburg Hamiltonian

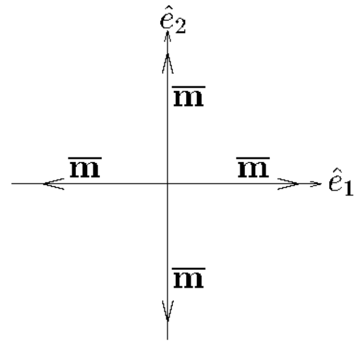
$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + v\sum_{i=1}^{n} m_i^4\right] \quad ,$$

for an n–component vector $\vec{m}(\mathbf{x}) = (m_1, m_2, \cdots, m_n)$. The "cubic anisotropy" term $\sum_{i=1}^{n} m_i^4$, breaks the full rotational symmetry and selects specific directions.

(a) For a fixed magnitude $|\vec{m}|$; what directions in the n component magnetization space are selected for $v > 0$ and for $v < 0$? What is the degeneracy of easy magnetization axes in each case?



$v > 0$ diagonal order



$v < 0$ cubic axis order

• In the figures below, we indicate the possible directions of the magnetization which are selected depending upon the sign of the coefficient v, for the simple case of $n = 2$:

This qualitative behavior can be generalized for an n-component vector: For $v > 0$, $\overline{\mathbf{m}}$ lies along the diagonals of a n-dimensional hypercube, which can be labelled as

$$\overline{\mathbf{m}} = \frac{\overline{m}}{\sqrt{n}}(\pm 1, \pm 1, \ldots, \pm 1),$$

and are consequently 2^n-fold degenerate. Conversely, for $v < 0$, $\overline{\mathbf{m}}$ can point along any of the cubic axes $\hat{e}_i$, yielding

$$\overline{\mathbf{m}} = \pm \overline{m}\hat{e}_i,$$

which is $2n$-fold degenerate.

(b) What are the restrictions on u and v for $\beta\mathcal{H}$ to have finite minima? Sketch these regions of stability in the (u, v) plane.

• The Landau-Ginzburg Hamiltonian for each of these cases evaluates to

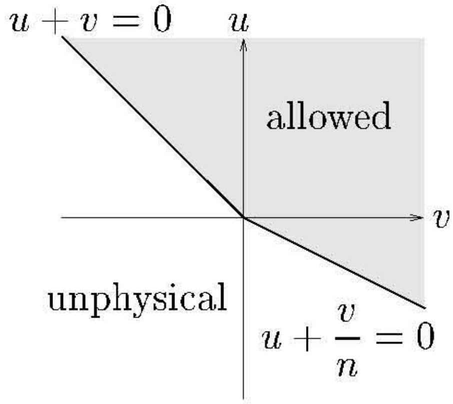
$$\begin{cases} \beta\mathcal{H} = \dfrac{t}{2}\overline{m}^2 + u\overline{m}^4 + \dfrac{v}{n}\overline{m}^4, & \text{if} \quad v > 0 \\ \beta\mathcal{H} = \dfrac{t}{2}\overline{m}^2 + u\overline{m}^4 + v\overline{m}^4, & \text{if} \quad v < 0 \end{cases},$$

implying that there are finite minima provided that

$$\begin{cases} u + \dfrac{v}{n} > 0, & \text{for} \quad v > 0, \\ u + v > 0, & \text{for} \quad v < 0. \end{cases}$$

Above, we represent schematically the distinct regions in the (u, v) plane.

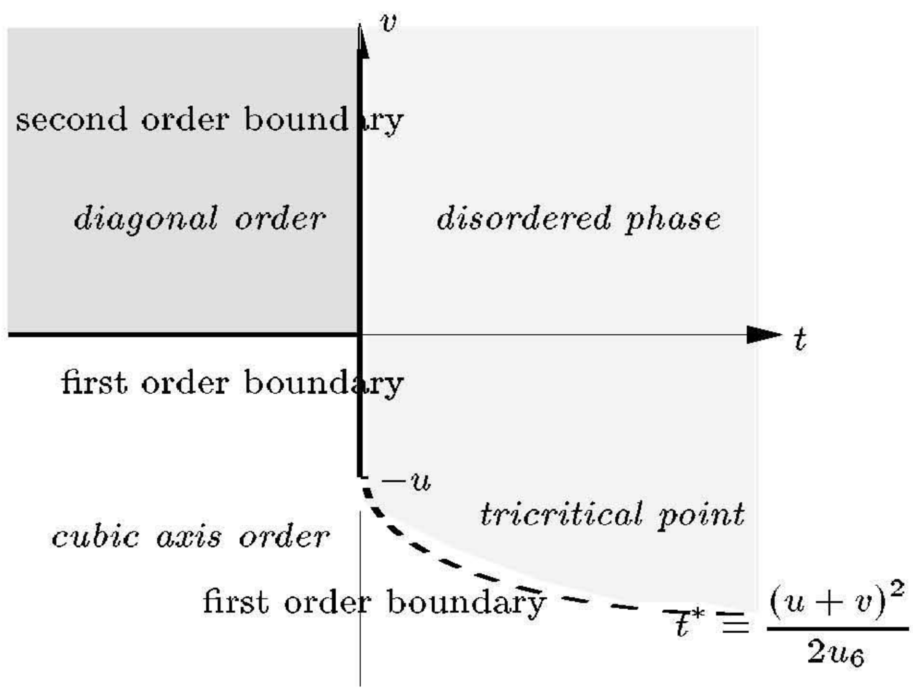
(c) In general, higher order terms (e.g. $u_6(\vec{m}^2)^3$ with $u_6 > 0$) are present and ensure stability in the regions not allowed in part (b); (as in case of the tricritical point discussed in earlier problems). With such terms in mind, sketch the saddle point phase diagram in the (t, v) plane for $u > 0$; clearly identifying the phases, and order of the transition lines.



• We need to take into account higher order terms to ensure stability in the regions not allowed in part b). There is a tricritical point which can be obtained after simultaneously solving the equations

$$\left.\begin{array}{l} t+4(u+v)m^2+6u_6m^4=0 \\ t+2(u+v)m^2+2u_6m^4=0 \end{array}\right\}, \quad \Longrightarrow \quad t^* = \frac{(u+v)^2}{2u_6}, \quad \overline{m}^2 = -\frac{(u+v)^2}{2u_6}.$$

The saddle point phase diagram in the (t, v) plane is then as follows:



(d) Are there any Goldstone modes in the ordered phases?

• There are no Goldstone modes in the ordered phases because the broken symmetry is discrete rather than continuous. We can easily calculate the estimated value of the transverse fluctuations in Fourier space as

$$\langle \phi_t(\mathbf{q})\phi_t(-\mathbf{q})\rangle = \frac{(2\pi)^d}{Kq^2 + \frac{vt}{u+v}},$$

from which we can see that indeed these modes become massless only when $v = 0$, i.e., when we retrieve the $O(n)$ symmetry.

****

6. *Cubic anisotropy– ε–expansion:*

(a) By looking at diagrams in a second order perturbation expansion in both u and v show that the recursion relations for these couplings are

$$\begin{cases} \dfrac{du}{d\ell} =& \varepsilon u - 4C\left[\ (n+8)u^2 + 6uv\ \right] \\ \dfrac{dv}{d\ell} =& \varepsilon v - 4C\left[\ 12uv + 9v^2\ \right] \end{cases},$$

where $C = K_d\Lambda^d/(t+K\Lambda^2)^2 \approx K_4/K^2$, is approximately a constant.

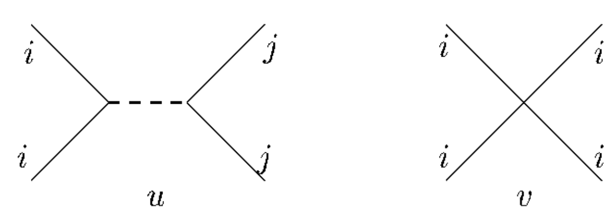
• Let us write the Hamiltonian in terms of Fourier modes

$$\begin{aligned}\beta\mathcal{H} =& \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{t+Kq^2}{2}\ \vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q}) \\ &+u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}}\ m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)\ m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \\ &+v\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}}\ m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)\ m_i(\mathbf{q}_3)m_i(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3),\end{aligned}$$

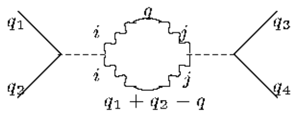
where, as usual, we assume summation over repeated indices. In analogy to problem set 6, after the three steps of the RG transformation, we obtain the renormalized parameters:

$$\begin{cases} t' =& b^{-d}z^2\tilde{t} \\ K' =& b^{-d-2}z^2\tilde{K} \\ u' =& b^{-3d}z^4\tilde{u} \\ v' =& b^{-3d}z^4\tilde{v} \end{cases},$$

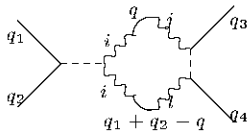
with $\tilde{t}$, $\tilde{K}$, $\tilde{u}$, and $\tilde{v}$, are the parameters in the coarse grained Hamiltonian. The dependence of $\tilde{u}$ and $\tilde{v}$, on the original parameters can be obtained by looking at diagrams in a second order perturbation expansion in both u and v. Let us introduce diagrammatic representations of u and v, as



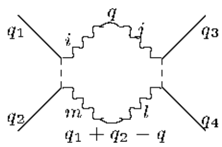
Contributions to u



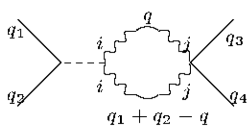
$$2\times 2\times 2n\frac{u^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$



$$2\times 2\times 4\times 2\frac{u^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

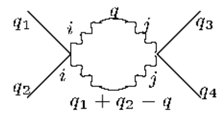


$$4\times 4\times 2\frac{u^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

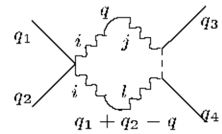


$$2\times 6\times 2uv\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

Contributions to v



$$6\times 6\times 2\frac{v^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$



$$6\times 4\times 2uv\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

where, again we have set $b = e^{\delta\ell}$. The new coarse grained parameters are

$$\begin{cases} \tilde{u} = u - 4C[(n+8)u^2 + 6uv]\delta\ell \\ \tilde{v} = v - 4C[9v^2 + 12uv]\delta\ell \end{cases},$$

which after introducing the parameter $\epsilon = 4 - d$, rescaling, and renormalizing, yield the recursion relations

$$\begin{cases} \dfrac{du}{d\ell} =\epsilon u - 4C[(n+8)u^2 + 6uv] \\ \dfrac{dv}{d\ell} =\epsilon v - 4C[9v^2 + 12uv] \end{cases}.$$

(b) Find all fixed points in the (u, v) plane, and draw the flow patterns for $n < 4$ and $n > 4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case.

• From the recursion relations, we can obtain the fixed points (u^*, v^*). For the sake of simplicity, from now on, we will refer to the rescaled quantities $u = 4Cu$, and $v = 4Cv$, in terms of which there are four fixed points located at

$$\begin{cases} u^* = v^* = 0 & \text{Gaussian fixed point} \\ u^* = 0 \quad v^* = \dfrac{\epsilon}{9} & \text{Ising fixed point} \\ u^* = \dfrac{\epsilon}{(n+8)} \quad v^* = 0 & \mathcal{O}(\text{n}) \text{ fixed point} \\ u^* = \dfrac{\epsilon}{3n} \quad v^* = \dfrac{\epsilon(n-4)}{9n} & \text{Cubic fixed point} \end{cases}.$$

Linearizing the recursion relations in the vicinity of the fixed point gives

$$A = \frac{d}{d\ell}\begin{pmatrix}\delta u\\ \delta v\end{pmatrix}_{u^*,v^*} = \begin{pmatrix}\epsilon - 2(n+8)u^* - 6v^* & -6u^*\\ -12v^* & \epsilon - 12u^* - 18v^*\end{pmatrix}\begin{pmatrix}\delta u\\ \delta v\end{pmatrix}.$$

As usual, a positive eigenvalue corresponds to an unstable direction, whereas negative ones correspond to stable directions. For each of the four fixed points, we obtain:

1. *Gaussian fixed point*: $\lambda_1 = \lambda_2 = \epsilon$, i.e., this fixed point is doubly unstable for $\epsilon > 0$, as

$$A = \begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}.$$

2. *Ising fixed point*: This fixed point has one stable and one unstable direction, as

$$A = \begin{pmatrix}\frac{\epsilon}{3} & 0\\ -4\frac{\epsilon}{3} & -\epsilon\end{pmatrix},$$

corresponding to $\lambda_1 = \epsilon/3$ and $\lambda_2 = -\epsilon$. Note that for $u = 0$, the system decouples into n noninteracting 1-component Ising spins.

3. *O(n) fixed point*: The matrix

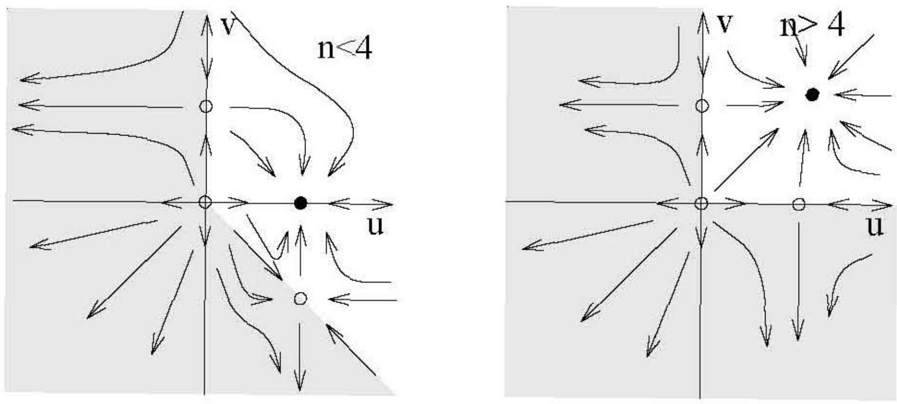
$$A = \begin{pmatrix}-\epsilon & -6\frac{\epsilon}{(n+8)}\\ 0 & \frac{(n-4)}{(n+8)}\epsilon\end{pmatrix},$$

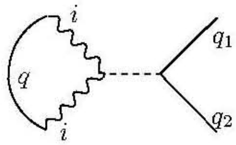
has eigenvalues $\lambda_1 = -\epsilon$, and $\lambda_2 = \epsilon(n-4)/(n+8)$. Hence for $n > 4$ this fixed point has one stable an one unstable direction, while for $n < 4$ both eigendirections are stable. This fixed point thus controls the critical behavior of the system for $n < 4$.

4. *Cubic fixed point*: The eigenvalues of

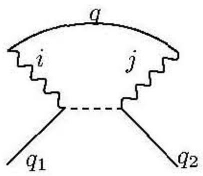
$$A = \begin{pmatrix}-\frac{(n+8)}{3} & -2\\ -4\frac{(n-4)}{3} & 4-n\end{pmatrix}\frac{\epsilon}{n},$$

are $\lambda_1 = \epsilon(4-n)/3n$, $\lambda_2 = -\epsilon$. Thus for $n < 4$, this fixed point has one stable and one unstable direction, and for $n > 4$ both eigendirections are stable. This fixed point controls critical behavior of the system for $n > 4$.

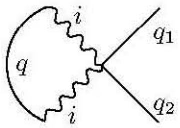




$$2nu\frac{K_d\Lambda^d}{(t+K\Lambda^2)}\delta l$$



$$4u\frac{K_d\Lambda^d}{(t+K\Lambda^2)}\delta l$$



$$6v\frac{K_d\Lambda^d}{(t+K\Lambda^2)}\delta l$$

$$\tilde{t} = t+4\frac{K_d\Lambda^d}{(t+K\Lambda^2)}[(n+2)u+3v]$$

$$\frac{dt}{dl} = 2t+4\frac{K_d\Lambda^d}{(t+K\Lambda^2)}[(n+2)u+3v]$$

In the (u,v) plane, $v^* = 0$ for $n < 4$, and the cubic term is irrelevant, i.e., fluctuations restore full rotational symmetry. For $n > 4$, v is relevant, resulting in the following flows:

(c) Find the recursion relation for the reduced temperature, t, and calculate the exponent ν at the stable fixed points for $n < 4$ and $n > 4$.

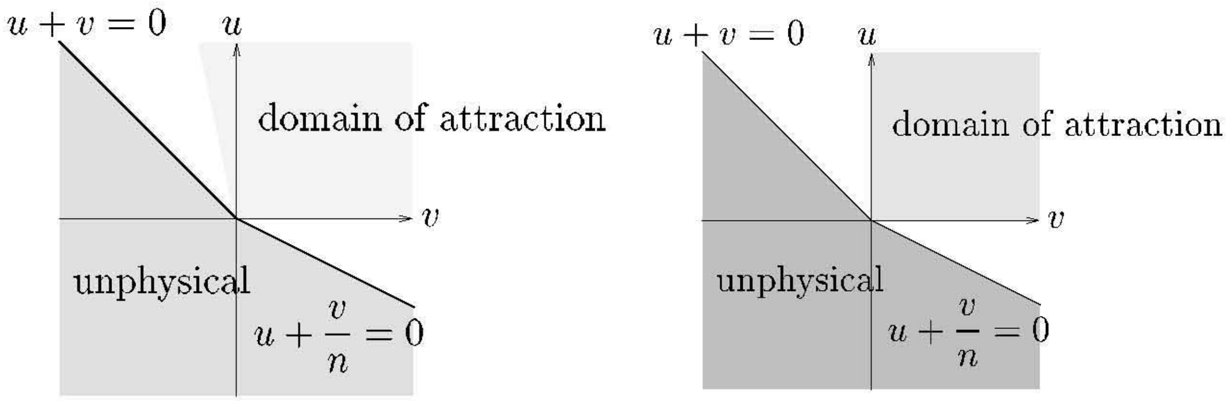
• Up to linear order in ϵ, the following diagrams contribute to the determination of $\tilde{t}$:

After linearizing in the vicinity of the stable fixed points, the exponent y_t is given by

$$y_t = 2 - 4C[(n+2)u^* + 3v^*], \Longrightarrow \nu = \frac{1}{y_t} = \begin{cases} \frac{1}{2} + \frac{(n+2)}{4(n+8)}\epsilon + \mathcal{O}(\epsilon^2) & \text{for} \quad n < 4 \\ \frac{1}{2} + \frac{(n-1)}{6n}\epsilon + \mathcal{O}(\epsilon^2) & \text{for} \quad n > 4 \end{cases}.$$

(d) Is the region of stability in the (u,v) plane calculated in part (b) of the previous problem enhanced or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about the nature of the phase transition for a small negative v and $n > 4$?

• All fixed points are located within the allowed region calculated in 1b). However, not all flows starting in classically stable regions are attracted to stable fixed point. If the RG flows take a point outside the region of stability, then fluctuations decrease the region of stability. The domains of attraction of the fixed points for $n < 4$ and $n > 4$ are indicated in the following figures:



$n < 4$ $\qquad$ $n > 4$

Flows which are not originally located within these domains of attraction flow into the unphysical regions. The coupling constants u and v become more negative. This is the signal of a fluctuation induced first order phase transition, by what is known as the Coleman–Weinberg mechanism. Fluctuations are responsible for the change of order of the transition in the regions of the (u,v) plane outside the domain of attraction of the stable fixed points.

(e) Draw schematic phase diagrams in the (t, v) plane ($u > 0$) for $n > 4$ and $n < 4$, identifying the ordered phases. Are there Goldstone modes in any of these phases close to the phase transition?

• From the recursion relation obtained in 2c) for the parameter t, we obtain the following non-trivial t^*

$$t^* = -\frac{1}{2}[(n+2)u^* + 3v^*] \propto -\epsilon$$

Therefore, the phase diagrams in the (t,v) plane is schematically represented as

As mentioned above, only when $n < 4$ fluctuations restore the full rotational symmetry. The parameter v is renormalized to zero, and there are Goldstone modes at the (u,v) plane, but only near the second order phase transition, where $K\xi_t^{-2} = tv/(u+v) \to 0$. In the ordered phases, the renormalized value of v is finite, albeit small, indicating that there are no Goldstone modes.

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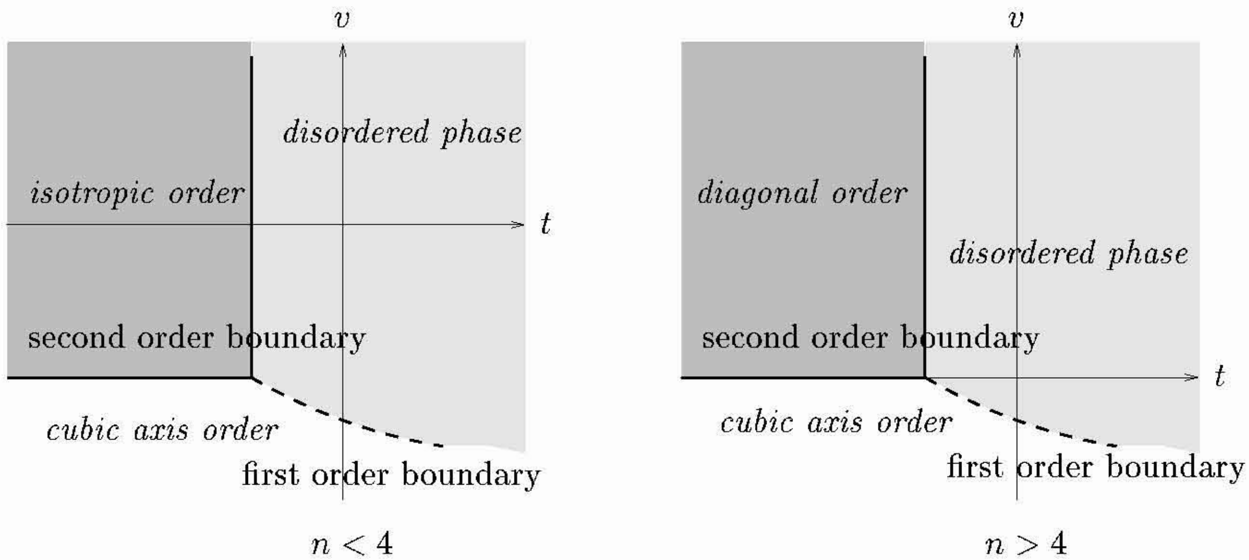
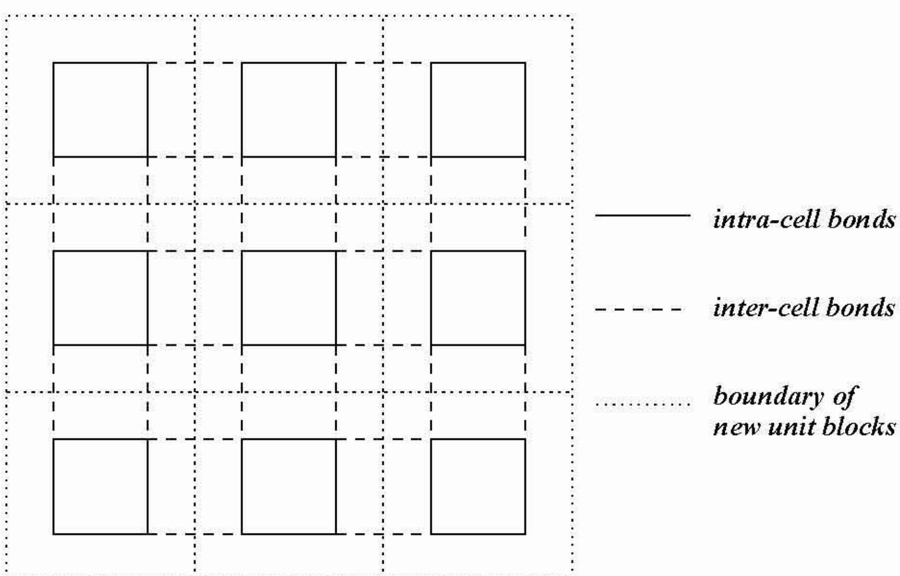
7. *Cumulant method:* Apply the Niemeijer–van Leeuwen first order cumulant expansion to the Ising model on a *square* lattice with $-\beta\mathcal{H} = K\sum_{<ij>}\sigma_i\sigma_j$, by following these steps:



(a) For an RG with $b = 2$, divide the bonds into *intra–cell* components $\beta\mathcal{H}_0$; and *inter–cell* components $\mathcal{U}$.

• The N sites of the square lattice are partitioned into $N/4$ cells as indicated in the figure below (the *intra-cell* and *inter-cell* bonds are represented by solid and dashed lines respectively).



The renormalized Hamiltonian $\beta\mathcal{H}'[\sigma'_\alpha]$ is calculated from

$$\beta\mathcal{H}'\left[\sigma'_\alpha\right] = -\ln Z_0\left[\sigma'_\alpha\right] + \langle\mathcal{U}\rangle_0 - \frac{1}{2}\left(\left\langle\mathcal{U}^2\right\rangle_0 - \langle\mathcal{U}\rangle_0^2\right) + \mathcal{O}\left(\mathcal{U}^3\right),$$

where $\langle\rangle_0$ indicates expectation values calculated with the weight $\exp(-\beta\mathcal{H}_0)$ at fixed $[\sigma'_\alpha]$.

(b) For each cell α, define a renormalized spin $\sigma'_\alpha = \text{sign}(\sigma^1_\alpha + \sigma^2_\alpha + \sigma^3_\alpha + \sigma^4_\alpha)$. This choice becomes ambiguous for configurations such that $\sum_{i=1}^{4} \sigma^i_\alpha = 0$. Distribute the weight of these configurations equally between $\sigma'_\alpha = +1$ and -1 (i.e. put a factor of $1/2$ in addition to the Boltzmann weight). Make a table for all possible configurations of a cell, the internal probability $\exp(-\beta\mathcal{H}_0)$, and the weights contributing to $\sigma'_\alpha = \pm 1$.

• The possible intracell configurations compatible with a renormalized spin $\sigma'_\alpha = \pm 1$, and their corresponding contributions to the intra-cell probability $\exp(-\beta\mathcal{H}_0)$, are given below,

$\sigma'_\alpha = 1$	Weight		Weight
$\begin{array}{\|cc\|} \hline + & + \\ + & + \\ \hline \end{array}$	e^{4K}	$\begin{array}{\|cc\|} \hline - & + \\ + & + \\ \hline \end{array}$	4×1
$\begin{array}{\|cc\|} \hline + & + \\ - & - \\ \hline \end{array}$	$4 \times 1 \times \dfrac{1}{2}$	$\begin{array}{\|cc\|} \hline + & - \\ - & + \\ \hline \end{array}$	$2 \times \mathrm{e}^{-4K} \times \dfrac{1}{2}$

$\sigma'_\alpha = -1$	Weight		Weight
$\begin{array}{\|cc\|} \hline - & - \\ - & - \\ \hline \end{array}$	e^{4K}	$\begin{array}{\|cc\|} \hline + & - \\ - & - \\ \hline \end{array}$	4×1
$\begin{array}{\|cc\|} \hline + & + \\ - & - \\ \hline \end{array}$	$4 \times 1 \times \dfrac{1}{2}$	$\begin{array}{\|cc\|} \hline + & - \\ - & + \\ \hline \end{array}$	$2 \times \mathrm{e}^{-4K} \times \dfrac{1}{2}$

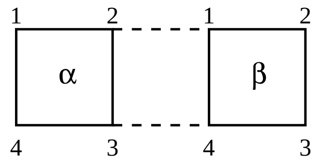
resulting in

$$Z_0\left[\sigma'_\alpha\right] = \prod_\alpha \left(e^{4K} + 6 + e^{-4K}\right) = \left(e^{4K} + 6 + e^{-4K}\right)^{N/4}.$$

(c) Express $\langle \mathcal{U} \rangle_0$ in terms of the cell spins σ'_α; and hence obtain the recursion relation $K'(K)$.

• The first cumulant of the interaction term is

$$-\langle \mathcal{U} \rangle_0 = K \sum_{\langle \alpha,\beta \rangle} \langle \sigma_{\alpha 2}\sigma_{\beta 1} + \sigma_{\alpha 3}\sigma_{\beta 4} \rangle_0 = 2K \sum_{\langle \alpha,\beta \rangle} \langle \sigma_{\alpha 2} \rangle_0 \langle \sigma_{\beta 1} \rangle_0,$$



where, for $\sigma'_\alpha = 1$,

$$\langle \sigma_{\alpha i} \rangle_0 = \frac{e^{4K} + (3-1) + 0 + 0}{(\mathrm{e}^{4K} + 6 + e^{-4K})} = \frac{e^{4K} + 2}{(e^{4K} + 6 + e^{-4K})}.$$

Clearly, for $\sigma'_\alpha = -1$ we obtain the same result with a global negative sign, and thus

$$\langle \sigma_{\alpha i} \rangle_0 = \sigma'_\alpha \frac{\mathrm{e}^{4K} + 2}{(e^{4K} + 6 + e^{-4K})}.$$

As a result,

$$-\beta \mathcal{H}' \left[\sigma'_\alpha \right] = \frac{N}{4} \ln \left(e^{4K} + 6 + e^{-4K} \right) + 2K \left(\frac{e^{4K} + 2}{e^{4K} + 6 + e^{-4K}} \right)^2 \sum_{\langle \alpha, \beta \rangle} \sigma'_\alpha \sigma'_\beta,$$

corresponding to the recursion relation $K'(K)$,

$$K' = 2K \left(\frac{e^{4K} + 2}{e^{4K} + 6 + \mathrm{e}^{-4K}} \right)^2.$$

(d) Find the fixed point K^*, and the thermal eigenvalue y_t.

• To find the fixed point with $K' = K = K^*$, we introduce the variable $x = e^{4K^*}$. Hence, we have to solve the equation

$$\frac{x+2}{x+6+x^{-1}} = \frac{1}{\sqrt{2}}, \qquad \text{or} \qquad \left(\sqrt{2} - 1 \right) x^2 - \left(6 - 2\sqrt{2} \right) x - 1 = 0,$$

whose only meaningful solution is $x \simeq 7.96$, resulting in $K^* \simeq 0.52$.

To obtain the thermal eigenvalue, let us linearize the recursion relation around this non-trivial fixed point,

$$\left. \frac{\partial K'}{\partial K} \right|_{K^*} = b^{y_t}, \quad \Rightarrow \quad 2^{y_t} = 1 + 8K^* \left[\frac{e^{4K^*}}{\mathrm{e}^{4K^*} + 2} - \frac{e^{4K^*} - e^{-4K^*}}{e^{4K^*} + 6 + \mathrm{e}^{-4K^*}} \right], \quad \Rightarrow \quad y_t \simeq 1.006.$$

(e) In the presence of a small magnetic field $h \sum_i \sigma_i$, find the recursion relation for h; and calculate the magnetic eigenvalue y_h at the fixed point.

• In the presence of a small magnetic field, we will have an extra contribution to the Hamiltonian

$$h \sum_{\alpha, i} \langle \sigma_{\alpha, i} \rangle_0 = 4h \frac{e^{4K} + 2}{(\mathrm{e}^{4K} + 6 + e^{-4K})} \sum_\alpha \sigma'_\alpha.$$

Therefore,

$$h' = 4h \frac{e^{4K} + 2}{(e^{4K} + 6 + e^{-4K})}.$$

(f) Compare K^*, y_t, and y_h to their exact values.

• The cumulant method gives a value of $K^* = 0.52$, while the critical point of the Ising model on a square lattice is located at $K_c \approx 0.44$. The exact values of y_t and y_h for the two dimensional Ising model are respectively 1 and 1.875, while the cumulant method yields $y_t \approx 1.006$ and $y_h \approx 1.5$. As in the case of a triangular lattice, y_h is lower than the exact result. Nevertheless, the thermal exponent y_t is fortuitously close to its exact value.

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8. *Migdal–Kadanoff method:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, on sites i of a hypercubic lattice, interacting with their nearest neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<ij>} \delta_{s_i, s_j} \ .$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Indentify all fixed points and note their stability.

• In $d = 1$, if we average over the q possible values of s_1, we obtain

$$\sum_{s_1=1}^{q} e^{K(\delta_{\sigma_1 s_1} + \delta_{s_1 \sigma_2})} = \begin{cases} q - 1 + e^{2K} & \text{if} \quad \sigma_1 = \sigma_2 \\ q - 2 + 2e^{K} & \text{if} \quad \sigma_1 \neq \sigma_2 \end{cases} = e^{g' + K' \delta_{\sigma_1 \sigma_2}},$$

from which we arrive at the exact recursion relations:

$$e^{K'} = \frac{q - 1 + e^{2K}}{q - 2 + 2e^{K}}, \qquad e^{g'} = q - 2 + 2e^{K}.$$

To find the fixed points we set $K' = K = K^*$. As in the previous problem, let us introduce the variable $x = e^{K^*}$. Hence, we have to solve the equation

$$x = \frac{q - 1 + x^2}{q - 2 + 2x}, \qquad \text{or} \qquad x^2 + (q-2)x - (q-1) = 0,$$

whose only meaningful solution is $x = 1$, resulting in $K^* = 0$. To check its stability, we consider $K \ll 1$, so that

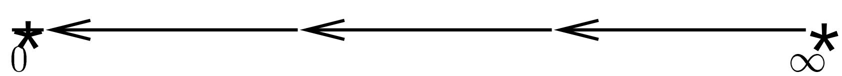
$$K' \simeq \ln\left(\frac{q + 2K + 2K^2}{q + 2K + K^2}\right) \simeq \frac{K^2}{q} \ll K,$$

which indicates that this fixed point is stable.

In addition, $K^* \to \infty$ is also a fixed point. If we consider $K \gg 1$,

$$e^{K'} \simeq \frac{1}{2} e^{K}, \quad \Longrightarrow \quad K' = K - \ln 2 < K,$$

which implies that this fixed point is unstable.



(b) Write down the recursion relation $K'(K)$ in d–dimensions for $b = 2$, using the Migdal–Kadanoff bond moving scheme.

• In the Migdal-Kadanoff approximation, moving bonds strengthens the remaining bonds by a factor 2^{d-1}. Therefore, in the decimated lattice we have

$$e^{K'} = \frac{q - 1 + e^{2\times 2^{d-1}K}}{q - 2 + 2e^{2^{d-1}K}}.$$

(c) By considering the stability of the fixed points at zero and infinite coupling, prove the existence of a non–trivial fixed point at finite K^* for $d > 1$.

• In the vicinity of the fixed point $K^* = 0$, i.e. for $K \ll 1$,

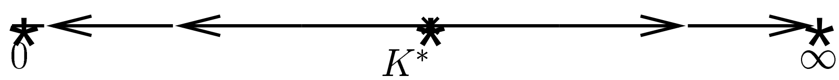
$$K' \simeq \frac{2^{2d-2}K^2}{q} \ll K,$$

and consequently, this point is again stable. However, for $K^* \to \infty$, we have

$$e^{K'} \simeq \frac{1}{2}\exp\left[\left(2^d - 2^{d-1}\right)K\right], \quad \Longrightarrow \quad K' = 2^{d-1}K - \ln 2 \gg K,$$

which implies that this fixed point is now stable provided that $d > 1$.

• As a result, there must be a finite K^* fixed point, which separates the flows to the other fixed points.



(d) For $d = 2$, obtain K^* and y_t, for $q = 3$, 1, and 0.

• Let us now discuss a few particular cases in $d = 2$. For instance, if we consider $q = 3$, the non-trivial fixed point is a solution of the equation

$$x = \frac{2 + x^4}{1 + 2x^2}, \qquad \text{or} \qquad x^4 - 2x^3 - x + 2 = (x - 2)(x^3 - 1) = 0,$$

which clearly yields a non-trivial fixed point at $K^* = \ln 2 \simeq 0.69$. The thermal exponent for this point

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} = 4\left[\frac{e^{4K^*}}{e^{4K^*} + 2} - \frac{e^{2K^*}}{1 + 2e^{2K^*}}\right] = \frac{16}{9}, \quad \Longrightarrow \quad y_t \simeq 0.83,$$

which can be compared to the exact values, $K^* = 1.005$, and $y_t = 1.2$.

By analytic continuation for $q \to 1$, we obtain

$$e^{K'} = \frac{e^{4K}}{-1 + 2e^{2K}}.$$

The non-trivial fixed point is a solution of the equation

$$x = \frac{x^4}{-1+2x^2}, \qquad \text{or} \qquad (x^3 - 2x^2 + 1) = (x-1)(x^2 - x - 1) = 0,$$

whose only non-trivial solution is $x = (1+\sqrt{5})/2 = 1.62$, resulting in $K^* = 0.48$. The thermal exponent for this point

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} = 4\left[1 - e^{-K^*}\right], \quad \Longrightarrow \quad y_t \simeq 0.61.$$

As discussed in the next problem set, the Potts model for $q \to 1$ can be mapped onto the problem of *bond percolation*, which despite being a purely geometrical phenomenon, shows many features completely analogous to those of a continuous thermal phase transition.

And finally for $q \to 0$, relevant to *lattice animals* (see PS#9), we obtain

$$e^{K'} = \frac{-1 + e^{4K}}{-2 + 2e^{2K}},$$

for which we have to solve the equation

$$x = \frac{-1+x^4}{-2+2x^2}, \qquad \text{or} \qquad x^4 - 2x^3 + 2x - 1 = (x-1)^3(x+1) = 0,$$

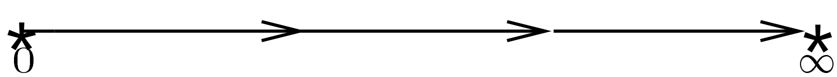
whose only finite solution is the trivial one, $x = 1$. For $q \to 0$, if $K \ll 1$, we obtain

$$K' \simeq K + \frac{K^2}{2} > K,$$

indicating that this fixed point is now unstable. Note that the first correction only indicates marginal stability ($y_t = 0$). Nevertheless, for $K^* \to \infty$, we have

$$e^{K'} \simeq \frac{1}{2}\exp[2K], \quad \Longrightarrow \quad K' = 2K - \ln 2 \gg K,$$

which implies that this fixed point is stable.



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9. *The Potts model:* The *transfer matrix* procedure can be extended to Potts model, where the spin s_i on each site takes q values $s_i = (1, 2, \cdots, q)$; and the Hamiltonian is $-\beta\mathcal{H} = K\sum_{i=1}^{N}\delta_{s_i,s_{i+1}} + K\delta_{s_N,s_1}$.

(a) Write down the transfer matrix and diagonalize it. Note that you do not have to solve a q^{th} order secular equation as it is easy to guess the eigenvectors from the symmetry of the matrix.

• The partition function is

$$Z = \sum_{\{s_i\}} < s_1|T|s_2 >< s_2|T|s_3 > \cdots < s_{N-1}|T|s_N >< s_N|T|s_1 >= \text{tr}(T^N),$$

where $< s_i|T|s_j >= \exp\left(K\delta_{s_i,s_j}\right)$ is a $q \times q$ transfer matrix. The diagonal elements of the matrix are e^K, while the off-diagonal elements are unity. The eigenvectors of the matrix are easily found by inspection. There is one eigenvectors with all elements equal; the corresponding eigenvalue is $\lambda_1 = e^K + q - 1$. There are also $(q-1)$ eigenvectors orthogonal to the first, i.e. the sum of whose elements is zero. This corresponding eigenvalues are degenerate and equal to $e^K - 1$. Thus

$$Z = \sum_\alpha \lambda_\alpha^N = \left(e^K + q - 1\right)^N + (q-1)\left(e^K - 1\right)^N.$$

(b) Calculate the free energy per site.

• Since the largest eigenvalue dominates for $N \gg 1$,

$$\frac{\ln Z}{N} = \ln\left(e^K + q - 1\right).$$

(c) Give the expression for the correlation length ξ (you don't need to provide a detailed derivation), and discuss its behavior as $T = 1/K \to 0$.

• Correlations decay as the ratio of the eigenvalues to the power of the separation. Hence the correlation length is

$$\xi = \left[\ln\left(\frac{\lambda_1}{\lambda_2}\right)\right]^{-1} = \left[\ln\left(\frac{e^K + q - 1}{e^K - 1}\right)\right]^{-1}.$$

In the limit of $K \to \infty$, expanding the above result gives

$$\xi \simeq \frac{e^K}{q} = \frac{1}{q}\exp\left(\frac{1}{T}\right).$$

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