PROFESSOR: Good afternoon. Usually, my first question is do you have any questions about the last lecture? We did this fly over over 100 equations to go from first principles to the [INAUDIBLE] Hamiltonian.

I've put up here for you the overview of this course. I won't go through it again. But I want to sort of make you aware that we started with the fundamental Hamiltonian for light atom interactions. But now we take a step back.

In the next few lectures, we simply talk about light. We talk about photons. We talk about the quantum nature of photons. And then, after understanding photons in more depth, we come back and look at more depths at atom photon interactions.

Let me just give you a pictorial summary of what I've just said. AMO physics is about atoms and photons. And in a cartoon like way, I can symbolize atoms by two level systems.

Or sometimes, we have a two level system which is coupled by tunneling that's also equivalent to a two level system. And then we have photons, two state systems. And photons are different.

Photons are not two state systems. They are actually harmonic oscillators. Usually, electromagnetic waves. But Hamiltonians for photons can be and have been realized by replacing the harmonic oscillator with the photon with vibrations, for example, of ions in an ion trap or micromechanical oscillators.

So cartoon picture again is photons in a cavity. But this is just another harmonic oscillator. So we have the atoms.

And most of what you should know about atoms has been discussed in 8.421. For

those of you who will take 8.421 next year, we'll have a lot of fun talking about the structure of atoms. Photons, well, I would hope a lot of it, at least the harmonic oscillator aspects, you took in 8.05 or in 8.321. And actually, starting today in this unit, we want to talk about the quantum age of light.

So you can say 8.421, we had a deep look at the atoms. We have now look at the photons. But eventually, what is very important is the interactions between the atoms and the photons.

That's not all. This is sort of a very well defined system. A two level system and one mode of the harmonic oscillator. But it will become very important-- and that's what we'll do in a few weeks-- to discuss what happens when this system-, the system above the dotted line, becomes an open system and we interact with the reservoir.

The reservoir will be relevant for relaxation. Atoms can decohere, can lose their decoherence. Photons will decay or decohere. Entanglement will get lost. And this phenomenon of relaxation requires the formalism of the master equation. And we will find the master equation in the form of the optical block equation.

If you want me to add in cartoon picture of the reservoir, I would add many, many harmonic oscillators. So reservoir is characterized by having many modes and having a capacity to absorb energy without changing its state.

So this is another kind of illustrated summary of what we will be doing in the next few weeks. And today, we want to talk about the quantum nature of light. We're actually talking mainly about already a very special form of light, namely monochromatic light in a single mode of the harmonic oscillator.

And there is still a lot to be discussed about the statistics, about squeezed states, about coherent states. We want to understand how the quantum state of light is modified, is transformed, by propagation, or how it's modified when the light runs through a beam splitter.

And all of this is already relevant for just a single mode. And then very soon, we'll also talk about two modes of light. Because we need two modes if we want to talk

about entanglement of two modes.

I hope you will realize in this discussion that the quantum nature of light is more than light is a wave in the form of electromagnetic wave, or light is a particle in the form of photons. Things go much, much deeper.

So the unit on quantum light will actually evolve in six stages. Today, we will be talking about photons and statistics, fluctuation, coherent states. On next week, on Monday, we want to talk about non-classical light, states which have no classical analogy in squeezed states.

Eventually, we want to focus on single photons. We will then talk about entangled photons. The fifth part is on interferometry, or more generally on metrology.

We will talk about the short noise limit and the fundamental Heisenberg limit of performing quantum measurements. And the last one is now opening up the light to interactions with atoms. So we will talk about interaction of one photon and one atom in this specific situation of cavity QED.

So this is an overview over the next, I don't know, three, four, five, six lectures. Today, we start with a very, very short review of the simple harmonic oscillator, just to set the stage. I then want to show you that in three minutes, I can discuss thermal light and the Planck Law by just writing down one or two equations in the formalism we have chosen.

So we cover thermal states. When we do laser physics, when we have light atom in the action, we always need coherent states. And we talk about quasi-probability distributions.

We then want to understand fluctuations, noise, and second order correlation functions. Probably this will spill over into the next class. We want to address some properties of the single photon. Which, as you will see, it seems a simple system, but it is the most non-classical system you can have, a single photon. But we'll come to that.

Let's start with very, very short review of the simple harmonic oscillator. It just also connects me with the last lecture. So how do we describe light in a single mode? And single mode means it is perfectly monochromatic.

Well, we know that-- and that's what we did last class-- that we have the Hamiltonian for the electromagnetic field, which is the quantum version of the electromagnetic energy, B square E square. And from that, we derived the quantized Hamiltonian. We introduced, just as a reminder, vector potentials, normal modes, and the normal mode operators became the a daggers. And in second quantization, this is the Hamiltonian for light, which is of course the Hamiltonian for harmonic oscillator.

What I need today is I have to rewrite the harmonic oscillator. And we have to look at two variables-- the momentum and the precision. Actually, the momentum will later, I will show you that the momentum of this harmonic oscillator is actually the operator of the electric field. So we need that in a second.

And of course, I assume you know from your knowledge of the harmonic oscillator that precision and momentum are linear combinations, symmetric and anti-symmetric, of a dagger a. And the unit is the prefactor square root of h bar omega over 2. Or here it is h bar over 2 omega.

And of course, in precision and momentum operators, the Hamiltonian is simply p square, should remind you of kinetic energy. Omega square, Q square should remind you of potential energy.

When we discuss light, we want to discuss the electric field. I had shown you in one of those 100 equations the operator for the electric field. The operator for the electric field was, of course, related to the vector potential.

And the vector potential, which is a normal mode expansion. And then we put everything backward. And we found that the electric field operator can actually be written.

Have to fill in a few things, but essentially, it's a minus a dagger. And a minus a

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dagger is nothing else than the momentum of the harmonic oscillator. So here, the momentum has a special role, because it is the operator of the electric field.

If you have light in a single mode, it has a harmonic oscillator representation. And the operator for the electric field is-- I put in a few bells and whistles in a moment-- but is in essence the momentum operator of the mechanical and analogous harmonic oscillator.

OK but the electric field has a polarization. We need that. We have the photon energy and the quantization volume. So what I'm writing down here is actually the electric field, this square root, is the electric field strings of a single photon in the quantized volume of our cavity. So v is the quantization volume.

Then if you have a single mode of the electromagnetic field, well, in contrast to a harmonic oscillator, we have a prorogation kr plus omega t. And here we have e to the minus i. I hope I get the signs wrong here. Minus omega t.

But, OK, we will mainly be looking at an atom at the origin. That's what we did in the dipole approximation. So when I said that the electric field operator corresponds to the momentum, I have to add now that it is at t equals 0 and at r equals 0. We'll come back to that in a few moments.

OK, since I've given you the operator-- let me just put the operator symbol on it-- the operator of the electric field, I have to say now one word about Heisenberg operators versus Schrodinger operators. I assume you're all familiar with the two different representations of quantum mechanics. We often use the Schrodinger representation where the operators are time independent. But in the Heisenberg representation, the wave function, the state vector, is time independent. And the time dependence of the evolution of the quantum system leads to a time dependence of the operator.

So what I've written down for you here is actually the Heisenberg operator, because it has a time revolution. But for a single mode, monochromatic electromagnetic field, the time evolution is just e to the i omega t. So if you want to obtain the Schrodinger

operator, you have to eliminate the time dependence by setting t equals 0.

At t equals 0, by definition, the Schrodinger operator and the Heisenberg operator are the same. But then, in the Schrodinger picture, the quantum state evolves and the operator stays put. In the Heisenberg picture, the quantum state stays constant and the operator evolves. So I'm usually writing down for you expressions for Heisenberg operators.

And if you simply set t equals 0 in this expression, you have the Schrodinger operator. Questions? Yes.

AUDIENCE: What happens when there's a factor of two that shows up in the electric field of a single photon, like the [INAUDIBLE]? I thought it was like-- [INAUDIBLE] it's just a factor of two [INAUDIBLE].

PROFESSOR: Yes, you have to be careful with factors of two. I think I got it right here. There is no extra factor of two. But the question is, OK, whenever you write down equations for the electric field, you want to sort of use a and a daggers photon number, or create photon numbers, so they are dimensionless.

So you always need a prefactor, which has a dimension of the electric field. And for the quantisation of light, you have two choices. One is I can take the electric field associated with the single photon. That's one option.

The other option is I take as a prefactor the electric field in the ground state, which you can say is the zero point fluctuations of the electric field and the two differ by a factor of two, because the energy of the vacuum field is 1/2 h bar omega, whereas 1 [INAUDIBLE] equals h bar omega. So that's one reason why you may sometimes find factors of two in some textbooks or not in others. But I think-- I checked. I think, as far as I know, it is consistent.

AUDIENCE: So these operators you've written down for e, this is the Heisenberg operator with the time frames right now?

PROFESSOR: This is the Heisenberg operator for the electric field. But it differs. I mean, at that

point, don't-- there's nothing really to worry about or get concerned about.

The question is, for monochromatic light, do we put the e to the i omega t into the state or the operator? It's really just an e to the i omega t factor has to be put either on this state or on the operator. These are the two choices.

Actually, my advice to you is don't even develop an intuitive conceptual understanding. And whether you are in the Schrodinger, try to sort of see the structure of the operator. I sometimes like the omega t, because it shows you how the electric field evolves. And you see that explicitly.

But in the end, it's only when you do real calculation that you have to think hard, OK, which picture am I using now? Which representation am I using now? I'm probably not 100% rigorous in this course when I write down quantum states and operators, whether I'm always in the Schrodinger picture or in the Heisenberg picture.

But by simply looking, does it have a time dependence? Yes. It's a Heisenberg picture. If I write down a quantum state, which has a time dependence, then I've chosen the Schrodinger representation. It's as easy as that. There was other? Yeah.

AUDIENCE: There's a sign wrong up there, because both a and a dagger goes into the i omega t as you have written down. So they should be kind of rotating right.

PROFESSOR: Give me one second. Actually, I just wanted to say I was careful with my notes, but I didn't carefully look at my notes. This is what I have in my notes. Does that satisfy you?

AUDIENCE: Yeah.

PROFESSOR: OK, good. OK so we want to talk about quantum states of light. The simplest eigenstates are, of course, the harmonic oscillator eigenstates. But, as I've already mentioned, and we'll elaborate on it further, these are actually non-classical states.

So quite often, we don't want those states. We want coherent states. But we need those number states said as one representation. So let me just introduce them.

So we'll label by n the states with n photons. And the energy of those states is n plus 1/2 h bar omega. The states are normalized.

Sometimes, out of laziness, because it doesn't matter, we can eliminate the 1/2. It just makes some equations more compact. And if I do that, I can write the Hamiltonian like this. You are familiar, of course, with commutators of a and a daggers. And the only non-trivial matrix element of a and a dagger that it raises and lowers the photon number by one.

OK, this was a short review of the simple harmonic oscillator as it applies to a single mode of the electromagnetic field. Now, just as a short example, in subsection two, we want to discuss the thermal state. And the thermal state is nothing else than photons in a black body cavity.

We have a photon field, which is in equilibrium at temperature t. So if you use the canonical ensemble, our statistical operator is given by this well known expression. And if I use-- I mean that's all.

I mean, isn't it wonderful once you're trained in statistical mechanics? This is all you have to know about a thermal state of light? That's it.

This is the Planck Law. All we have to do is take the Hamiltonian, which we have written down, upstairs. So this is e to the minus n photons. n times h bar omega is the energy over kt divided by the partition function. This is the density matrix, a statistical operator for single mode light at thermal equilibrium.

The partition function is very simple, because this sum over Boltzmann factors this is a geometric series, which can be simply summed up in this way-- divided by kt. So therefore, this expression here can be written as-- the density matrix in thermal equilibrium can be written in the number bases.

And it's diagonal. And the prefactor is the probability to have n photons populated in thermal equilibrium. And the probability for n photons is given here. We've evaluated the partition function.

So then if you put it together, we simply get this-- you can call it Boltz-Einesten factor, which tells us what is the probability to find an harmonic oscillator with n [INAUDIBLE]. And this is the expression-- e to the minus h bar omega.

So this result takes care of many situations-- actually, almost all situations-- you find in every day life unless you switch on a laser pointer. So this describes-- and resistive filament. It describes the light generated by gas discharges. It describes the light of the sun.

And this kind of thermal light, because it has the-- I mean, light in thermal equilibrium maximizing the entropy under the given boundary condition, it's also called chaotic light. It's just another name for thermal light.

Well, we should now lean back and take the attitude, if I know the statistical operator of an ensemble, I know everything about it. And every quantity I'm interested in can be calculated. So just as an example, we know what is a probability to find n photons. But now we want to ask what is the energy density, or what is the mean excitation number. What is the average of n?

So for that, we have to take the number operator, a dagger a, and perform the trace with the product of the statistical operator. And this is, of course, nothing else when we look for the probability to find n photons. And this probability is weighted with n. And what we get is the very simple result.

It's a Bose-Einstein distribution law with a chemical potential of 0, which is the familiar distribution. We can now use the quantity n to rewrite our previous expression for the probability to find n photons in thermal equilibrium. It is n bar to the power n divided by 1 over n bar times n plus 1.

So all this should be very familiar to you. And I'm not doing the final step, but I could immediately get Planck's Law now. All I have to use it sum up now. This is a single mode of light, single monochromatic light. All I have to do is to get Planck's law is to sum overall modes using the density of modes.

Any question? We will not be talking a lot in this course about chaotic and thermal light, because we are mainly interested in the purer form how atom interacts with light. And this will be when light is in a more specific state, not in a thermal ensemble, in a more specific state which are number states or coherent states.

But nevertheless, I feel if I don't tell you a little bit about ordinary light, you will not be able to appreciate quantum light. So I want to use this example now of a thermal distribution of light and write down for you in the next 90 seconds expressions for the fluctuations. So you should know a little bit about what is the statistic of thermal light.

If you find n photons in a black cavity and you look again, it's the standard deviation of the photon number square root. And is it Poissonian, or is it much larger? So you should know about that, because it will then make a difference when we talk about the purer states of light, coherent states, and number states, squeeze states, and all that.

OK, so we want to look now at fluctuations. Let me just first mention that we have normalized correctly the probabilities. Just one second.

What we are interested now is the fluctuations. So we want to know the variance. We want to know what is delta n squared. And delta n squared is of course n square average minus n average squared. And we want to find out what is that for thermal light, for chaotic light.

So what we need now is we need an expression. We have already an expression up there for n average. And now we need an expression for n square average.

We can get that as follows. If we take the probability, multiply it with n-- n minus 1. Sum over.

So I have a brain cramp. This equation is in my notes, but I don't-- maybe step back-- I don't see where it comes from. But from this expression, of course, we have now an expression for n average squared. n squared-- the sum over it.

Just a second. n average-- n thermal-- n average squared. Anyway, I don't want to waste class time. It made sense to me when I wrote the notes. I just can't reproduce the argument.

So this should be obvious. It's not obvious to me right now. But let's just take it and run with it.

The left hand side is n square minus n multiplied with the probability. So let me just take a note. I will show you where it comes from on Monday. But assuming that it's true, and I know it is true, that actually contains n square average minus n average is 2n average squared.

So therefore, we have now our expression for the average of n square, which is 2n average squared plus n average. And therefore, for thermal light, for chaotic light, the variance, delta n square, is n average square plus n average.

So the question is if this is our variance-- the variance of thermal light, of chaotic light, is that. What you should see immediately is that it is much more fluctuating then Poissonian statistics. Well, if you compare now with Poissonian statistics, Poissonian statistics would have fluctuations where, well, the RMS fluctuations are square root n. The variance is just n average. So thermal light is super Poissonian, has much, much stronger fluctuations than Poissonian distribution.

OK, one question for you. If you look at the distribution of photons-- and we have written it down here. Let me just scroll up. Well, you probably don't have an auto-plot function in your brain.

But if you plot this function, and I'm asking you what is the most probable photon number you will find for light in thermal equilibrium?

AUDIENCE: Zero.

PROFESSOR: Zero. It is a broad Gaussian centered at zero. So you have a broad Gaussian. The average photon number is related to the width of the Gaussian.

But the nature of thermal light is that whenever you look into a black body cavity and

ask, what is the most probable photon number for any mode, it is actually zero. No matter how hot the cavity is and how many photons you have in it, the photon number distribution is always a broad Gaussian with a maximum at zero.

We'll come back to that later When we talk about coherent states. Then you will appreciate the difference. But I will be referring back to the result we have just derived for the photon number distribution.

Let me just take a note. The most probably n is n equals 0. OK, I hope with that-- and this is III, the third subsection-- you can really appreciate the properties of coherent states.

Coherent states were introduced by Roy Glauber in 1963. It's actually something you should wonder about. Coherent states seem so natural. I mean, we teach it in all courses. And when we think about harmonic oscillator, we often think in terms of coherent states.

But they were not immediately used and invented when quantum mechanics was invented in the '30s. It really took somebody like Roy Glauber to invent coherent states. And that happened in the '60s.

Maybe the reason why people didn't immediately jump at coherent states is because coherent states are not eigenstates of an Hamiltonian. The definition of a coherent state is actually an eigenstate, not of an Hamiltonian. It's an eigenstate of the annihilation operator.

And of course, the annihilation operator is non-Hermitian. So it's a strange operator to look for eigenstates. But this is what defines the coherent state, namely that e acting on a coherent state alpha equals alpha times alpha.

I'm using here the standard notation that alpha is both the label for the coherent state written down here, and is also the eigenvalue of the coherent state when acted upon with the annihilation operator. We normalize those states. And since we're talking about eigenvalues of a non-Hermitian operator, alpha can be any

complex number.

OK, so now we have introduced a second set of basis functions for light. Remember, we had the basis functions n, the eigenfunctions of the harmonic oscillator, and now we've defined a new set of eigenfunctions, the coherent state alphas. Of course, the next obvious question is how are the two representations connected?

So what we want to do now is we want to figure out how do we write a coherent state alpha in the n basis in the basis of eigenfunctions of the harmonic oscillator? So we want to write-- this is just-- want to expand the coherent state into eigenfunctions n. So the coefficients we want to calculate are those matrix elements which are defined as cn.

So what we have to use is in order to find this representation, we have to use the fact that alpha is an eigenstate of the annihilation operator. So we know that this is alpha times alpha. But we can also let the annihilation operator a, we can let the annihilation operator a act on this state and in this sum.

And the annihilation operator, of course, does what it is supposed to do. It lowers the photon number by one. And we get the matrix element square root n. And then by just changing the summation index by one, we can write it as n times cn plus 1 times the square root of n plus 1.

So this is when the annihilation operator acts on the sum. And now, I'll continue with the lower line, which is alpha cn times n. So what I've found now is I've found two ways to hide the same state, namely a acting on alpha. And therefore, those coefficients have to be the same.

And what I just indicated in blue is a recursion relation, which expresses cn plus 1 by cn. So every time you go from cn to cn plus 1, you multiply with the square root of n plus 1. So if you stare at it for a moment, you find that cn is related to c0.

Yes, cn plus 1 by alpha n over n factorial. So of course, we get only the relation between the coefficients. But if you throw in that we want to normalize the state, this

determines c0. And then, we get as a final result the normalized representation of the coherent state in terms of number states.

Excuse me, I think the square root is missing here, because it appears here times n. OK, so what we have found here is now how coherent states are written down in the number state basis.

So now we know coherent states. And we can ask a number of questions. So what is this?

So what did we define? And what have we now derived? Let's look at some of the properties.

The first one is that if you look at this expression, this is the amplitude that will have n photons in a coherent state. The square of the amplitude is the probability. So the probability to find n photons is the square of the amplitude above. So this is that. And this is now a Poissonian distribution.

So it's immediately clear that those coherent states have much, much smaller fluctuations in the photon number than the thermal state we discussed before. Because this is Poissonian, whereas the thermal state had a variance which was much, much larger than Poissonian. OK, that's number one.

Number two is we would like to know what is the photon number. So we want to know what is the expectation value for n. That's actually very easy to find. You could, of course, get the average number of photons in by evaluating the expansion of the coherent state in terms of number states.

But sometimes, you should also try to maybe directly evaluate the coherent state. And the photon number operator is a dagger a. And now you see here we are lucky, because a acting on alpha is just getting alpha.

But a dagger acting on alpha on the left hand side gives us alpha star, the complex conjugate. So what we obtain here is simply the absolute value of alpha square. And this is our expression for the average photon number.

I have already mentioned that the variance must be-- the variance must be n, because it's a Poissonian distribution. But since it is so nice to have creation annihilation operators acting on coherent states, let's just calculate-- let's verify that this distribution-- we know that distribution is Poissonian, but let's verify directly that the variance of the photon number in the coherent state is n, is Poissonian.

If you need the variance, we need an average. And we need n square average. Let's just calculate n square average.

So we need now the operator for n square, which is a dagger a, a dagger a. We know already if those two operators act on the [INAUDIBLE], that this just gives us alpha star alpha. And then we are left with a a dagger.

Now, a dagger acts on alpha. We don't know what a dagger is acting on alpha. We know that alpha is an eigenstate of a, but not of a dagger. So therefore, we want to change the order of a and a dagger, use a commutator.

And then of course, a dagger can act here. A can act here. And we just get another factor of alpha. So therefore, in a very straightforward way, we find now that n square is alpha square times alpha square plus 1. And using the result for the mean photon number, this is nothing else than n bar, 1 plus n bar.

So if we now put it together, we want to know the variance. The variance is n square average, which we've just calculated, minus n average squared. And hooray, we find n bar. And we verify again that the fluctuations are Poissonian.

What else can we ask? We've asked about the statistics of fluctuation. We've asked about the probability to find photons. Well, what else do you want to know about light? Electric field.

Oh, that's something I forgot to ask. I will come back to that later. But just sort of to connect it, what is the average electric field for thermal light? Zero. And what else can it be?

Because any value of the electric field, whether the electric field is positive or

negative, requires that the-- I say it loosely-- that the photon has a phase. The phase will define whether it's positive or negative. But all different phases of the photon field, all different-- what I mean now is coherent states, but have different phases-- they are all degenerate.

So in a thermal ensemble, the states are only populated due to their energy, they're populated by Boltzmann factors. And therefore, there is no distinction what the phase of the state are. So therefore, we haven't defined yet what the phase of a photon is, so therefore I'm a little bit guarded with my language.

But in other words, whatever the state with the phase is in a thermal ensemble, you have all phases simultaneously. And therefore, you never have an expectation value of e. But of course, we had an average number of photons.

And that means we have an expectation value of e square, because the expectation value of e square is the expectation value of the energy. The state has energy. But it doesn't have an expectation value for the electric field.

OK, so maybe with that motivation, we want to figure out now does this coherent state, which we have constructed, does that have now an expectation value for the electric field? So there is one thing you of course know about this electric field. Whenever we have an electric field, it needs an amplitude and a phase.

And what I just mentioned to you is that in a thermal ensemble, we were just looking at the partition function and such. We didn't even mention a phase. There was not any complex number in any of the equations we wrote down about thermal light. No complex number, no phase, means no electric field.

But this is different now for the coherent state. Because I mentioned that alpha in general can be complex, the annihilation operator can have complex eigenvalues. So therefore, in the coherent state-- because it is characterized by a complex number-- we have now the possibility to include a phase. And what I want to show you is, in the next few minutes, is that yes indeed, the coherent state has an expectation value of the electric field.

Instead of simply using the equation for the electric field operator-- remember electric field operator was a minus a dagger-- I want to introduce something else which will turn out to be very, very useful. And one of the application of this concept is to show you what the electric field is.

This is sort of a visual representation, a visual representation of the states of the electromagnetic field. So I want to sort of give you, in a two dimensional diagram, a visual representation of thermal states, coherent states, number states. And this visual representation will make it immediately obvious what the electric field is.

So the visual representation will represent what are called quasi-probabilities. So I have to define for you now those quasi-probabilities. So I denote the quasi-probability by Q. And I want to know what is the quasi-probability Q for any statistical operator which describes a light.

It could be the operator for thermal light. It could be statistical operator also includes, of course, pure states. And the pure state could be a coherence state. So this is a way to define our quasi-probabilities for any statistical operator which is describing the light.

And at the quasi-probabilities, well, the probability for what? Well, the probabilities for alpha. And alpha is related to the coherent state.

So in other words, we say a statistical operator has a quasi-probability with a coherent state alpha by simply calculating this diagonal matrix element of the statistical operator with alpha. Well, that's an abstract definition, but it's exact. What we should now do is I want to show you some examples. And then you will actually see that it's actually a wonderful definition.

So my first example is let's look at the quasi-probability for the vacuum state. So if the statistical operator is simply the ground state of the harmonic oscillator, no photon, no nothing, just empty vacuum. So the quasi-probability for the statistical operator is now a pure state.

So I can say, what is a quasi-probability of the vacuum state to be in state alpha? Well, following the definition above, it's nothing else than the matrix element, the overlap, between the vacuum and the coherent state alpha. I've given you a path, the representation of the coherent state alpha in terms of number states. And all you have to do is look now upstairs, what is the amplitude that this representation includes a vacuum state.

So it's just read of the amplitude c0 from the state above. And this was simply e to the minus alpha squared. So therefore, if I-- those quasi-probabilities can be plotted in a two dimensional plane, because it's a function of alpha.

So if I use the real part of alpha, the imaginary part of alpha, now I can plot the quasi-probability Q. And since it's hard for me to do a 3-D plot on the tablet, I just sort of shade the region. Black means the amplitude is large. And now it's sort of here. It means the amplitude falls off.

So what I'm plotting here is a Gaussian, a Gaussian with the widths of [INAUDIBLE] with the width of 1/2. I think in the standard deviation of the Gaussian, you have a factor of 2. Anyway, it's on the order of-- the Gaussian width on the order of [INAUDIBLE] here, I think it's 1 over square root 2, or whatever the definition of the standard deviation is. So here we have a Gaussian. Any question?

So for pure state, we're simply asking what is the amplitude of the purse state to overlap to be in a coherent state. And what we are plotting is nothing else than the probability that this state is a coherent state with value alpha. You had a question?

AUDIENCE: [INAUDIBLE].

PROFESSOR: The next example, and we have already prepared for that, is the thermal state. Example number two is the thermal state. So we want to know what is the quasi-probability for the thermal state as a function of alpha.

The statistical operator for the thermal states, we had derived [INAUDIBLE] was the probability to find n photons. And in the number representation, the statistical operator is diagonal. And then the quasi-probabilities are nothing else, following the

definition above, then those probabilities times the matrix element, the overlap between alpha and n.

And if you look up what we had derived for the probability pn, you find a very simple result. What we obtain is again a Gaussian. It is also centered at the origin.

So therefore, since I didn't label my axis, it looks exactly as a vacuum state. It's just that the Gaussian is much, much broader. And I will come to that, but an intuitive reading of this Gaussian is, well, this Gaussian has a peak at the origin.

At the origin, this is where alpha equals 0. And this is directly related to the question I asked you before, that the photon number probability is peaked at n equals 0. But I hope this will become even clearer when we move on to the third example, which is now Q of alpha of a coherent state.

Maybe just before I even discuss the quasi-probability of a coherent state, I have to address one thing up front. What would you expect the quasi-probability of a coherent state to be?

AUDIENCE: The Gaussian [INAUDIBLE].

PROFESSOR: Pretty good. But I said before, the quasi-probability is-- let me just show you how it was defined. Here was my definition of the quasi-probability. And you can see the quasi-probability is the diagram matrix element of the statistical operator with a coherent state.

But if our statistical operator is a coherent state, what would you expect this probability to be? Just naively.

AUDIENCE: One.

PROFESSOR: One or a delta function. I mean, it's sort of-- we ask, we ask-- if you expand this statistical operator in coherent states, what do we get? But if this operator is a coherent state, let's say it's a coherent state beta, you would expect, well, then we only get something non-vanishing if alpha equals beta.

So you would naively expect that what you get is a delta function. The coherent state beta has a quasi-probability Q of alpha, which is peaked at beta. But it's peaked as a delta function. Now, I'm calculating it for you right now, and the result is this is not the case.

The reason is-- and I will show that to you-- is that the coherent states are not your ordinary basis function. The coherent state forms a basis which is over complete. So we have sort of more coherent states than necessary.

There is some redundancy. And therefore, the coherent state alpha and the coherent state beta, if there's only a small difference between alpha and beta, they have overlap. Coherent states which differ by only a little bit in their eigenvalue, they're not orthogonal.

So this is a complication. It's one of the complications we have to deal with when we want to describe coherent states and when we want to describe laser lights. But this is one of the properties.

So therefore, when I'm now discussing with you is what are the quasi-probabilities of a coherent state, it will not be a delta function. But we should simply follow the definition, and everything will fall into place.

So our statistical operator for a pure coherent state is, of course-- this is a statistical operator with pure state beta. And the quasi-probability for the coherent state beta to be an alpha is this matrix element squared. And you can calculate that, for instance, by just putting in alpha, alpha and beta, the expansion of alpha and beta in number states.

So just have to solve the integral and calculate it. And what you find is not a delta function, but a decaying exponential. So you can say, as long as alpha and beta do not differ by more than one, there is substantial overlap.

So in other words, what we get for this quasi-probability is something which is centered at beta. But it is again a Gaussian. So the important message here is coherent states are not orthogonal to each other. They form an over complete

basis.

Maybe as a side remark, it may come as a surprise to you, because you have been trained too much in basis states of emission operators, of energy eigenstates. And those energy eigenstates form a complete basis. The coherent states are eigenfunctions of very strange operator, a non-emission annihilation operator. So therefore, the few things you took for granted do not apply here.

So we have our nice diagram with-- [INAUDIBLE] diagram for quasi-probabilities. So now if you have-- that was the real part of alpha. That was the imaginary part of alpha.

So if you take that as a complex plane, and I'm drawing now as a phase of the complex number beta, which is the eigenvalue of the coherent state beta, then the quasi-probability is centered near beta. And it's a decaying Gaussian. I'm not showing it in here explicitly, but it is simply the vacuum state we discussed before displaced by beta.

My fourth example, just to show you. So far, I've shown you that everything is Gaussian. Thermal light is a broad Gaussian, coherent state is a narrow Gaussian, the vacuum state is a narrow Gaussian. Let me just show you something which is non-Gaussian, which is the number state.

If you put into the definition of the quasi-probability the number state, and you calculate the quasi-probability as a function of alpha, then you find that what you get is actually not a Gaussian. What you get is a ring, a ring with a certain width. So this is the representation of a number state. And the radius of this ring is proportional to the square root of the number of photons in a number state.

We'll come back to photon states in much more detail very soon. But I just wanted to show you that at this point. So quasi-probabilities are not always Gaussian. They really depict something, they show us something important about-- they show us important differences about the quantum states of light.

Any questions? Yes.

AUDIENCE: [INAUDIBLE] a delta function [INAUDIBLE] radius, [INAUDIBLE] little past [INAUDIBLE]?

PROFESSOR: Can you hold the question for three minutes? OK. They quick answer is it's not. But in three minutes, I want to tell you that there are three possible definitions of quasi-probabilities-- the Q function, to W function, and the P function. In one of the functions, it's a delta function. In the Q function, it's not. But we'll come to that.

But let's for now stick to the Q function. The next thing we want to put in is the time dependence. I want to show you that when we have quantum states of light-- and right now, we've just shown the quasi-probability distribution at a snapshot of t equals 0. What happens as a function of time?

What I want to show you is-- and this makes those quasi-probabilities also nice and intuitive-- that as a function of time, the quasi-probability distribution simply rotates with an angular frequency of omega. So let me show that to you.

We know that we have a coherent state. And what we want to understand is what happens when we act on the coherent state with the time propagation operator, which is the Hamiltonian in the exponent. Well, in order to evaluate the left hand side, we can simply take the expansion of coherent states into in a number basis, because we know that the time evolution of a number state is simply the energy of the number state with which it's given by the energy of the number state, which is n times h bar omega.

So now if you look for a second at this expression, you realize that we can absorb the time evolution if you redefine alpha to become alpha times e to the i omega t. So in other words, the time evolution of the coherent state alpha preserves the character as a coherent state. It just means we get an incoherent state whose eigenvalue is now alpha times e to the i omega t.

So that means the following. If you want to look at the time evolution in terms of quasi-probabilities, we have our diagram with the real part of alpha, with the

imaginary part of alpha. And let's assume we had a coherent state, which happened-- this is sort of now my circle.

It's, you can say, a high contrast representation of the Gaussian. So this is the quasi-probability of the initial state alpha. But as time propagates, it just gets multiplied with e to the i omega t. That means when the time evolution is just displacing the state over there, and as time goes by, the coherent state is just moving in a circle. And after one full period of omega, we are back where we started.

Now let me give you now an intuitive picture what the electric field is. I will be a little bit more exact in two or three minutes. So we have a diagram of the real part of alpha and the imaginary part of alpha.

For a harmonic oscillator, I can also label it as x and p. You know, in harmonic oscillator, when you can think about it classically, if you have some initial distribution of a mechanical oscillator in x and p, what the system actually does is it simply rotates on a circle in phase space. The classical phase space is x and p.

And I haven't really told you that I was-- I didn't want to get lost in complex definitions. I defined for you the quasi-probability, but the quasi-probability is a generalization of the classical phase space function. Phase space function is a function of p and x. So take my word for a second, and allow me to identify the real axis with x and the imaginary axis with p.

Well, then p is the electric field. So now, what you should visually take from those pictures is that if you want to know what the electric field is, you just sort of project this fuzzy ball on the imaginary axis. So in other words, when I ask you what is now the electric field as a function of time, at t equals 0, I project this onto the p axis.

And what I get is something which is centered at 0. But it has also a certain fuzziness. The fuzziness is given by the size of the disk, or by the width of the Gaussian.

And if I now use this picture as a function of time, as this fuzzy ball rotates around,

the electric field goes up and down in one cycle. And the fuzziness moves with it. So the fuzziness here for the electric field is related-- and that's what we want to discuss next week-- is related to short noise.

I also want to tell you that the coherent state is, to some extent, the best possible way to define an electric field. It's a minimum uncertainty state. I want to show you that the coherent state is sort of at the minimum of Heisenberg's Uncertainty Relation.

So I think if you take this picture, you will immediately realize that you draw the quasi-probability, you project on the vertical axis. And you get the electric field that tells you immediately if you have a thermal state, which is a Gaussian centered here. You project it. You get an electric field 0. It has an enormous fuzziness, but it's 0.

And this picture that everything rotates with angular frequency omega also tells you, as expected, that in a thermal state, it's already a circularly symmetric distribution. When it rotates, nothing changes.

AUDIENCE: [INAUDIBLE], couldn't you find, eigenstates for that operator itself, and then get rid of this fuzziness?

AUDIENCE: Could it just be, like, eigenstates of the harmonic oscillator, so not stationary at all?

PROFESSOR: I'm a little bit confused now. Actually, two people mind if I teach a few minutes longer? I can just finish the-- I want to have one, I have one footnote. I really want to sort of give it to you, because I think it will help some of you. Pardon?

AUDIENCE: We have a class starting at 2:30.

PROFESSOR: There's a class starting at 2:30?

AUDIENCE: Yes.

PROFESSOR: Then I have to wrap up. Sorry. Then let me just tell you what I will do at the beginning of the next class. What I've told you here with the projection of the electric

field, this is not 100% rigorous for the quasi-probabilities Q.

Due to the non-commutativity of operators, when we generalize phase space function into quantum mechanics, we can order operators in a symmetric way, in a normal way, in an anti- normal way. And therefore, there are three different quasi-probabilities.

The Q probability is the easiest to define. It comes in very handy. But what I showed you with the projection does not rigorously apply to the Q distribution. It applies to the p or w distribution.

But the difference is, unless you really go into subtleties, are minor. So intuitively, what I did is correct. And to answer your question, an eigenstate of the electric field, I have a little bit problems with that. Because if you take this projection, what you need is you need something which is very, very narrow.

But since this area is in Heisenberg uncertainty limited area, it means delta x delta p equals h bar over 2. The only way how you can create this electric field is you can completely squeeze it. So to the best of my knowledge, an electric field eigenstate would be a completely strongly squeezed state.

We'll talk about that next week. But then you realize, if you've completed squeezed your state after a quarter period, this infinitely squeezed ellipse is now standing upside down. And after a quarter period your electric field is completely uncertain.

So I think you can define an operator, you can find a state which has a sharp value of the electric field at one moment of time. But then it rotates around. And what was a highly certain electric field becomes highly uncertain later.

OK, enough things to discuss. We'll go on next week.