8.512 Theory of Solids II
Spring 2009

8.512 Theory of Solids **Problem Set 12** **Due May 13, 2009**

Consider a Fermi gas with dispersion ϵ_k and a repulsive interaction $U\delta(\boldsymbol{r})$. Now if $N(0)U > 1$, we find in mean field theory the spontaneous appearance of the order parameter:

$$\Delta = U\langle n_\uparrow - n_\downarrow \rangle \quad ,$$

and the splitting of the up- and down-spin bands, so that the mean field Hamiltonian is

$$H_{\text{MF}} = \sum_k \left(\tilde{\epsilon}_{k\uparrow} c^\dagger_{k\uparrow} c_{k\uparrow} + \tilde{\epsilon}_{k\downarrow} c^\dagger_{k\downarrow} c_{k\downarrow} \right) \quad ,$$

where

$$\begin{aligned} \tilde{\epsilon}_{k\uparrow} &= \epsilon_k - \Delta/2 \\ \tilde{\epsilon}_{k\downarrow} &= \epsilon_k + \Delta/2 \quad . \end{aligned}$$

1. Consider the **transverse** spin susceptibility. Instead of $\chi_x = dM_x/dH_x$, where $\boldsymbol{M} = -2\mu_B(\boldsymbol{\sigma}/2)$ it is more convenient to consider the response to H_+ which couples to a spin flip excitation, i.e., $\chi_\perp = d(M_+/dH_+)$ where $M_+ = \frac{1}{2}(M_x + iM_y)$ and $H_+ = H_x + iH_y$. Show that the response function for the mean field Hamiltonian is given by $\chi^0_\perp = \mu_B^2 \Gamma_0$ where

   $$\Gamma_0(q,\omega) = \sum_k \frac{f\left(\tilde{\epsilon}_{k+q,\downarrow}\right) - f\left(\tilde{\epsilon}_{k,\uparrow}\right)}{\omega - \tilde{\epsilon}_{k+q\downarrow} + \tilde{\epsilon}_{kq\uparrow} + i\eta} \quad .$$

   This is the generalization of the Lindhard function to a spin split band.

2. Now include the interaction term in the response to the transverse field in a self consistent field approximation. Show that

$$\chi_\perp(q,\omega) = \frac{\mu_B^2 \Gamma_0(q,\omega)}{1 - U\Gamma_0(q,\omega)} \quad .$$

3. The poles of the numerator in $\chi_\perp$ describe the single particle-hole excitations. Sketch the region in (ω, q) space where $Im\chi_\perp \neq 0$ due to these excitations.

4. The other pole in $\chi_\perp(q,\omega)$ occurs when the denominator vanishes. Calculate the dispersion of this pole which we identify as the spin wave excitation as follows:

   (a) Show that at $q = \omega = 0$, the denominator vanishes. [Hint: the condition $1 - U\Gamma_0 = 0$ is the same as the self-consistency equation for $\Delta(T)$.]

   (b) Expand $\Gamma_0(q,\omega)$ for small q, ω and show that the location of the pole of $\chi_\perp$ is given by $\omega(q) = Cq^2$. Note that unlike the Lindhard function for free fermions, the existence of the gap Δ makes the expansion well behaved.