Holography Duality (8.821/8.871) Fall 2014
Assignment 2

Sept. 27, 2014 **Due Thursday, Oct. 9, 2014**

- Please remember to put **your name** at the top of your paper.

Note:

- The four laws of black hole mechanics were originally formulated in

  Bardeen, Carter and Hawking, "The Four Laws of Black Hole Mechanics", Commun. math. Phys. 31, 161-170 (1973).

- For a a recent review of black hole uniqueness theorems, see

  P. Mazur, "Black Uniqueness Theorems", hep-th/0101012.

- For reviews of the large N expansion of gauge theories, see

  - S. Coleman, "1/N", in *Aspect of Symmetry*, Cambridge Univ. Press, (1985).
    Highly recommend you to read p.368-p.378. Coleman's discussion is very elegant and transparent.

  - J. Maldacena, "TASI 2003 Lectures on AdS/CFT", arXiv:hep-th/0309246. See Sec. 2. This is a rather short discussion, but summarizes all the important elements.

  - A. Manohar, "Large N QCD", arXiv:hep-ph/9802419.
    The discussion by Manohar contains pedagogic introduction to all the basic ingredients and also extensive discussion of applications to QCD.

Problem Set 2

1. **Hawking temperature from analytic continuation (10 points)**

   Consider a general class of black hole metrics which can be written as

   $$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + \cdots \tag{1}$$

   where $f(r)$ and $h(r)$ have a first order zero at the horizon $r = r_0$ and the other part of metric represented by $\cdots$ is regular there (or does not vanish). Show that the Hawking temperature of such a black hole is given by

   $$T_H = \frac{\sqrt{f'(r_0)h'(r_0)}}{4\pi}. \tag{2}$$

2. **Kerr-Newman metric (25 points)**

   A most general black hole in asymptotically flat spacetime is characterized by its mass M, angular momentum J and charge Q. Such a metric is called Kerr-Newman metric, which can be written as

   $$ds^2 = -\frac{\rho^2 \Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{3}$$

   where

   $$\rho^2 = r^2 + a^2\cos\theta^2, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad a \equiv \frac{J}{M} \tag{4}$$

   $$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \quad \omega = \frac{a}{\Sigma}(r^2 + a^2 - \Delta) \tag{5}$$

   In the above expressions, for notational simplicities, I have set $G_N = 1$. There is also an electric potential, which I have suppressed.

   (a) Show that the Hawking temperature temperature T_H, the horizon value r_+ of the radial coordinate r, the horizon area A, and the angular velocity Ω at the black hole horizon can be written as

   $$\begin{aligned} r_+ &= M + \sqrt{M^2 - a^2 - Q^2} & (6) \\ A &= 4\pi(r_+^2 + a^2) & (7) \\ T_H &= \frac{2(r_+ - M)}{A} & (8) \\ \Omega &= \frac{4\pi a}{A} & (9) \end{aligned}$$

   Note that none of the above should require extensive calculation and to calculate the temperature it is simplest to use (2).

(b) Note that in order for r_+ to be real we should have

$$M^2 \geq a^2 + Q^2 . \tag{10}$$

A black hole which saturates the inequality is called extremal.[1] Show that an extremal black hole has zero temperature, but nonzero entropy. Calculate its entropy. Note that the black hole third law does not prevent a nonzero zero-temperature entropy and thus is less stringent than the third law of thermodynamics.

(c) Now set $J = 0$ and consider an extremal charged black hole with $M = Q$. Write down the metric for this special case explicitly and show that the horizon lies at an infinite proper distance away. Show that near the horizon, the geometry reduces to $\text{AdS}_2 \times \mathbb{R}^2$, where AdS_2 is a two-dimensional anti-de Sitter spacetime. (For a nonextremal black hole, the near horizon region is always Rindler.)

3. **Laws of black hole Mechanics (30 points)**

Consider a Kerr-Newman black hole in asymptotically flat spacetime characterized by mass M, angular momentum J and charge Q. In addition to (6)–(9), the electric potential Φ of the black hole horizon can be written as

$$\Phi = \frac{4\pi Q r_+}{A} \tag{11}$$

(a) Using (6)–(9) and (11), verify the first law of black hole mechanics. (Again this should not require extensive calculations.)

(b) Consider a rotating black hole with mass M and angular momentum J, but $Q = 0$. Suppose that such a black hole loses energy and all of its angular momentum by some *classical process*. Using the second law of black hole mechanics to show that at most 29% of the initial mass can be radiated away.

(c) Consider a nonextremal charged black hole with $J = 0$, i.e. $M > Q$. We want to reduce the temperature of the black hole to zero by throwing particles at it. Clearly in order to reach extremal limit the particles we throw in should have their charges q greater than their mass m. Also in order for a charged particle to be able to fall into the black hole, the gravitational attraction should be no smaller than the repulsive electric force, which implies that

$$qQ \leq Mm . \tag{12}$$

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[1]A solution for which (10) is violated does not have a horizon. Instead it has a naked singularity (i.e. a singularity not hidden behind an event horizon). You should convince yourself of this, although I will not ask you to show it.

The two conditions thus lead to

$$1 < \frac{q}{m} \leq \frac{M}{Q} \tag{13}$$

The above expression implies that as the extremal limit is approached, the window of particles one can use becomes smaller and smaller. Show that using this procedure it takes an infinite number of steps of reach the extremal limit.

4. **A gas of radiation and maximal entropy bound (20 points)**

Consider a gas of massless particles confined inside a spherical box of radius R. You can assume that the particles are all scalars, non-interacting and there are Z species of them.

(a) Express the entropy of the system in terms of the total energy E.

(b) Now consider the energy of the gas is such that the system is on the verge of forming a black hole of size R. Find the ratio of the entropy of such a gas to the entropy of a black hole of size R. You will find that the ratio is always much smaller than 1 as far as R is greater than the Planck length l_p (which is about $10^{-33}cm$).

(c) Now let us keep the energy density ρ of the system fixed (e.g. by keeping T fixed) and vary the system size R. What is the maximal size R_M the system can have before it collapses into a black hole? Find the ratio of the entropy of such a gas with size R_M to the entropy of a black hole of the same size. You will find the ratio is much smaller than 1 as far as the energy density $\rho \ll m_p^4$, where m_p is the Planck mass.

Of course at the energy density of order m_p^4 we would expect quantum gravitational effects to be in full display and could not really say anything about such a system.

(d) Show that for both (b) and (c), you can always dial the number of species Z big enough so that the entropy of the gas becomes larger than that of a black hole and thus violates the maximal entropy bound. Find the minimal value of Z for which this happens for both (b) and (c).

Clearly such a value will depend on the size or energy density of the system and is astronomically big for an ordinary system (i.e. energy density not close to Planck density).

5. **Wilson loop in the large N limit (15 points)**

Consider a Wilson loop in an $SU(N)$ gauge theory along some closed loop C,

$$W_R(C) = \left\langle \operatorname{Tr} P \exp\left(ig \oint_C A_\mu dx^\mu \right) \right\rangle \tag{14}$$

where $A_\mu = A_\mu^a T_R^a$ with T_R^a $SU(N)$ generators in a representation R and $\langle \cdots \rangle$ denotes the expectation value in some state (say vacuum or thermal state).

(a) In the large N limit, find the relation between $W_F(C)$ and $W_A(C)$ where subscript F and A denote the fundamental and adjoint representations respectively.

(b) Now consider a confining theory. Remind yourself (or find out) how a Wilson loop should behave in such a theory. What does (a) imply about the string tension for a fundamental and an adjoint quark? Try to give an intuitive explanation for the result.

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