Massachusetts Institute of Technology
Department of Physics

Physics 8.942 Fall 2001

Problem Set #1
Due in class Tuesday, September 18, 2001.

1. **Conserved Momenta**
Show the following: If $\partial_\alpha g_{\mu\nu} = 0$ for all $\{\mu, \nu\}$, then P_α is conserved along a geodesic $x^\mu(\lambda)$, where $P_\alpha \equiv g_{\alpha\beta} dx^\beta / d\lambda$.

2. **Robertson-Walker Metric**
Consider the general Robertson-Walker metric, written in the form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right] . \tag{i}$$

Note that for $k > 0$ the complete spacetime has two copies of the domain $0 \le r \le k^{-1/2}$, just as a unit sphere has two copies of the cylindrical coordinate range $0 \le \sqrt{x^2 + y^2} \le 1$ (the northern and southern hemispheres).

Find coordinate transformations that will put the line element in the following forms:

$$ds^2 = a^2(\tau) \left[-d\tau^2 + d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2) \right] , \tag{ii}$$

$$ds^2 = a^2(\tau) \left[-d\tau^2 + \frac{d\bar{r}^2 + \bar{r}^2 (d\theta^2 + \sin^2\theta \, d\phi^2)}{\left(1 + \frac{1}{4} k \bar{r}^2\right)^2} \right] , \tag{iii}$$

$$ds^2 = -dt^2 + a^2(t) \left[1 + \frac{k}{4} \left(x^2 + y^2 + z^2 \right) \right]^{-2} \left(dx^2 + dy^2 + dz^2 \right) . \tag{iv}$$

For each case, indicate the full range of the variables. Give explicit formulae for $r(\chi)$ and $\bar{r}(r)$. (Hint: Different forms may be required for $k > 0$, $k < 0$, and $k = 0$. Note also that $a(\tau)$ is not the same function of its argument as $a(t)$.)

3. **Cosmological and Doppler Redshifts**
Consider an object with radial coordinate χ_e in a Robertson-Walker spacetime (using the form ii given in Problem 2). The object emits a burst of nearly monochromatic radiation at time τ_e with wavelength λ_e in its own rest frame. A fundamental (comoving) observer is at $\chi = 0$ with 4-velocity $\vec{V}_o = a^{-1} \vec{e}_\tau$. The observer detects the radiation at time τ_o with wavelength λ_o. The redshift is defined as $z \equiv (\lambda_o / \lambda_e) - 1$.

a) Assume that the emitter is comoving (i.e. at fixed spatial coordinates) so that its four-velocity is $\vec{V}_e = a_e^{-1}\vec{e}_\tau$ where $a_e \equiv a(\tau_e)$. Evaluate the redshift in terms of the expansion scale factor $a(\tau)$.

b) Now suppose that the emitter is no longer comoving. Instead, it has a radial "peculiar" velocity component v_r, which is the radial three-velocity measured by a *comoving observer at* χ_e. (In other words, v_r is the radial velocity component in an orthonormal basis fixed at χ_e.) What are the emitter's four-velocity components V_e^τ and V_e^χ in terms of v_r and a_e? Show that your result makes sense in the non-cosmological limit $a(\tau) =$ constant. (Do not assume $v_r^2 \ll 1$.)

c) Continuing part b), what is the object's redshift as seen by the observer? Show that if $a(\tau) =$ constant, you recover the radial Doppler shift formula of special relativity while if $v_r = 0$ you recover part a).

d) Now suppose that the emitter also has a tangential velocity (relative to the comoving frame) with orthonormal components v_θ and v_ϕ, i.e. the peculiar velocity has arbitrary direction. Show that the redshift is given by

$$1 + z = \frac{a_o}{a_e}\frac{1 + v_r}{\sqrt{1 - v^2}} \ .$$

e) (Bonus challenge): Suppose that the observer at $\chi = 0$ is no longer comoving but has a three-velocity $\underline{v}_o$ relative to the comoving frame. How is $1 + z$ modified from part d)?

4. **An Empty Universe**
For a $k = -1$ Robertson-Walker spacetime with $\rho = p = 0$ show from the Friedmann and energy conservation equations that the line element becomes

$$ds^2 = -dt^2 + t^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \ .$$

Find an explicit coordinate transformation to show that this metric describes Minkowski spacetime.