PROBLEM SET 5
Post date: Thursday, March 16th
Due date: Thursday, March 23rd

1. Space garbage

In a convenient coordinate system, the spacetime of the earth is approximately

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \\
&= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right)\left(dx^2 + dy^2 + dz^2\right),
\end{aligned}
$$

where M is the earth's mass. In the second version, we've remapped the spherical coordinates to Cartesian coordinates in the usual way:

$$
x = \sin\theta\cos\phi\,, \qquad y = \sin\theta\sin\phi\,, \qquad z = \cos\theta\,.
$$

Note that the Cartesian form of the spacetime metric is conveniently written $g_{\alpha\beta} = \eta_{\alpha\beta} + 2\Phi\mathrm{diag}(1,1,1,1)$, with $\Phi \equiv GM/r$. You may assume $\Phi \ll 1$ throughout this problem.

The space shuttle orbits the earth in a circular $(u^r = 0)$, equatorial $(\theta = \pi/2, u^\theta = 0)$ orbit of radius R.

(a) Using the geodesic equation, show that an orbit which begins equatorial remains equatorial: $du^\theta/dt = 0$ if $u^\theta = 0$ and $\theta = \pi/2$ at $t = 0$. (Hint: begin by computing the non-zero connection coefficients. Use the results of Carroll 3.3.)

We now require that the orbit must remain circular: $du^r/d\tau = 0$.

(b) By enforcing this condition with the geodesic equation, derive an expression for the orbital frequency

$$
\Omega \equiv \frac{d\phi/d\tau}{dt/d\tau}\,.
$$

Does the result look familiar?

The next part is most conveniently described in Cartesian coordinates; you may describe the shuttle's orbit as $x = R\cos\Omega t$, $y = R\sin\Omega t$.

An astronaut releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x^i_{\mathrm{garbage}} - x^i_{\mathrm{shuttle}}$.

(c) Using the equation of geodesic deviation, work out differential equations for the evolution of ξ^x, ξ^y, and ξ^z as a function of time. You may neglect terms in $(GM/r)^2$, and you may treat all orbital velocities as non-relativistic. You will need the Cartesian connection coefficients for this.

(d) Suppose the initial displacement is $\xi^x = \xi^y = 0$, $\xi^z = L$, $d\xi^i/d\tau = 0$. Has the astronaut succeeded in getting rid of the garbage?

2. Light bending

The spacetime of the sun can be written using the same line element as that of Problem 1 (substituting the sun's mass for M). Consider a light ray which initially is moving purely along the x axis. Suppose that it passes near the sun with impact parameter (the distance between the axis passing through the sun's center and the axis along which the light initially moves) b, as shown in this figure:
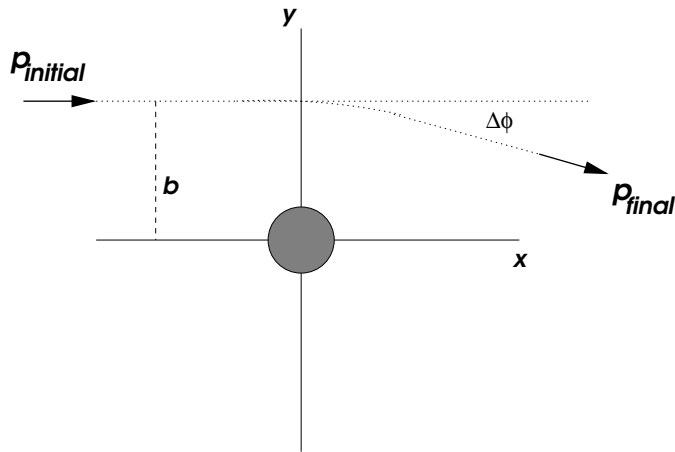
Using the geodesic equation, compute the angle by which the sun bends the light ray, $\Delta\phi = (p^y/p^x)_{\text{final}}$, expressing your answer using GM and b. You may assume that $GM/r \ll 1$ and that the bending angle is very small (so that $p^x_{\text{final}} \simeq p^x_{\text{initial}}$ to leading order in $\Delta\phi$).

Historical note: One can crudely calculate this effect in Newtonian theory by imagining that a photon has an effective mass $m_{\text{eff}} = \hbar\omega/c^2$ and asking what is the transverse momentum deflection imparted by Newtonian gravity. The answer one finds is exactly half the angle computed using the geodesic equation applied to the weak-field spacetime. The factor of two arises because of "spatial curvature" — the factor $(1 + 2\Phi)$ that multiplies the spatial sector of the spacetime metric. An expedition to observe an eclipse in Africa led by the British astrophysicist Arthur S. Eddington confirmed the relativistic prediction by carefully measuring stellar positions near the limb of the sun and comparing to their positions when the sun is in a different part of the sky. These results were instrumental in convincing the scientific world that general relativity was a valid description of gravity, and were largely responsible for Einstein becoming a household name.

It's worth noting that the expedition's data were not of particularly good quality; it's something of a miracle that they were able to get such good agreement with the relativistic prediction. Modern measurements using radar pulse propagation in the solar system measure this bending angle with very high precision; the relativistic prediction is typically confirmed with a fractional precision $\sim 10^{-5}$.

3. **Parallel transport on a sphere**

   On the surface of a 2-sphere of radius a, $ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$. Consider the vector $A_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e, over the range $0 \leq \phi \leq 2\pi$ at $\theta = \theta_0$). What is the resulting vector $\vec{A}$? What is its magnitude $(\vec{A} \cdot \vec{A})^{1/2}$? (Hint: derive differential equations for A^θ and A^ϕ as a function of ϕ.)

4. **Curvature of a sphere**

   (a) Compute all the nonvanishing components of the Riemann tensor R_{ijkl} $[(i,j,k,l) \in (\theta, \phi)]$ for the surface of a 2-sphere.

   (b) Consider the parallel transport of a tangent vector $\vec{A} = A^\theta \vec{e}_\theta + A^\phi \vec{e}_\phi$ on the sphere around an infinitesimal parallelogram of sides $\vec{e}_\theta \, d\theta$ and $\vec{e}_\phi \, d\phi$. Using the results of part (a), show that to first order in $d\Omega \equiv \sin\theta \, d\theta \, d\phi$, the length of $\vec{A}$ is unchanged, but its direction rotates through an angle equal to $d\Omega$.

   (c) Show that, if $\vec{A}$ is parallel transported around the boundary of any simply connected solid angle Ω, its direction rotates through an angle Ω. ("Simply connected" is a topological term meaning that the boundary of the region could be shrunk to a point; it tells us that there are no holes in the manifold or other pathologies. See http://mathworld.wolfram.com/SimplyConnected.html for illustrations.) Compare with the result of Problem 3.

5. **Killing vectors and curvature**

   Prove that the relations

   $$\begin{aligned} \nabla_\nu \nabla_\mu \xi^\alpha &= R^\alpha{}_{\mu\nu\beta} \xi^\beta \\ \Box \xi^\alpha &\equiv \nabla^\mu \nabla_\mu \xi^\alpha = -R^\alpha{}_\beta \xi^\beta \end{aligned}$$

   are satisfied by any Killing vector ξ^α. This result is not too difficult to derive using the commutator $[\nabla_\alpha, \nabla_\beta] = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha$ and the identities

   $$\begin{aligned} \left[\nabla_\alpha, \nabla_\beta\right] V^\mu &= R^\mu{}_{\nu\alpha\beta} V^\nu \\ \left[\nabla_\alpha, \nabla_\beta\right] V_\mu &= -R^\nu{}_{\mu\alpha\beta} V_\nu \end{aligned}$$

   (If you are reading Schutz, these identities are incorrectly given on p. 171 — the minus sign is missing in the second identity.)

6. **Riemann tensor for 1+1 static spacetimes**

   (a) Compute all the nonvanishing components of the Riemann tensor for the spacetime with line element

   $$ds^2 = -e^{2\phi(x)} dt^2 + e^{-2\psi(x)} dx^2 \ .$$

   (b) For the case $\phi = \psi = \frac{1}{2} \ln|g(x - x_0)|$ where g and x_0 are constants, show that the spacetime is flat and find a coordinate transformation to globally flat coordinates $(\bar{t}, \bar{x})$ such that $ds^2 = -d\bar{t}^2 + d\bar{x}^2$.