## 15.053/8 February 21, 2013

# Simplex Method Continued

# Quote of the Day

**"Everyone designs who devises courses of action aimed at changing existing situations into preferred ones."**

**-- Herbert Simon**

# Today's Lecture

- Very quick review of the simplex algorithm.
- Phase 1:  How to obtain the initial bfs
- Finiteness (assuming bases do not repeat)
  - Degeneracy
  - Anti-cycling rule(s)
- Alternative optima

# A very quick review

| -z | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4.5 | 6 | 0 | 0 | 0 | | 0 |
| 0 | 6 | 5 | 8 | 1 | 0 | 0 | | 60 |
| 0 | 10 | 20 | 10 | 0 | 1 | 0 | | 150 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | | 8 |

**The *basic variables* here are -z, $s_1$, $s_2$, $s_3$.**

**It is optional whether to call -z basic.**

**The basic feasible solution (bfs) is**

**$z = 0$; $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $s_1 = 60$; $s_2 = 150$; $s_3 = 8$**

# A very quick review

| $-z$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | | RHS | Ratio |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4.5 | 6 | 0 | 0 | 0 | | 0 | |
| 0 | 6 | 5 | 8 | 1 | 0 | 0 | | 60 | 60/6 |
| 0 | 10 | 20 | 10 | 0 | 1 | 0 | | 150 | 150/10 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | | 8 | 8/1 |

**If all reduced costs are ≤ 0, then you are optimal. Otherwise, choose a reduced cost that is positive.**

**We could have chosen the 5 or the 4.5 or the 6.**

**Use the min ratio rule to determine the pivot element (and the exiting variable).**

# Ending conditions: Optimality

**If all coefficients in the z-row are nonpositive ($\bar{c}_i \leq 0$ for all i), then the current basic solution is optimal.**

| Basic Variable | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| -z | 1 | 0 | -5 | 0 | 0 | -1 | = | -1 |
| $x_3$ | 0 | 0 | 2 | 1 | 0 | -2 | = | 1 |
| $x_4$ | 0 | 0 | -1 | 0 | 1 | -2 | = | 7 |
| $x_1$ | 0 | 1 | 6 | 0 | 0 | 0 | = | 3 |

# Ending conditions: Unboundedness

**If the z-row coefficient of $x_s$ is positive for some s, and if all (other) coefficients in the column for $x_s$ are nonpositive, then the optimal objective value is unbounded from above.**

| BV | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| -z | 1 | 0 | -2 | 0 | 0 | +1 | = | -6 |
| $x_3$ | 0 | 0 | 2 | 1 | 0 | -2 | = | 4 |
| $x_4$ | 0 | 0 | -1 | 0 | 1 | -2 | = | 2 |
| $x_1$ | 0 | 1 | 6 | 0 | 0 | 0 | = | 3 |

$z = 6 + \otimes$
$x_1 = 3$
$x_2 = 0$
$x_3 = 4 + 2\otimes$
$x_4 = 2 + 2\otimes$
$x_5 = \otimes$

# The pivot rule (min ratio version)

| Basic Var | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| -z | 1 | 0 | -2 | 0 | 0 | 6 | = | -11 | $= -z_0$ |
| $x_3$ | 0 | 0 | 2 | 1 | 0 | 2 | = | 4 | |
| $x_4$ | 0 | 0 | -1 | 0 | 1 | -2 | = | 1 | |
| $x_1$ | 0 | 1 | 6 | 0 | 0 | 3 | = | 9 | |

**Choose a variable $x_s$ (column) for which the z-row coefficient is positive.**

**Determine the constraint for which the following ratio is minimum. {RHS coeff / Col coeff : Col coeff > 0}**

| | | | |
|---|---|---|---|
| Constraint | (1) | (2) | (3) |
| Ratio | 4/2 | -2 < 0 | 9/3 |

# The pivot

| Basic Var | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| -z | 1 | 0 | -2 | 0 | 0 | 6 | = | -11 |
| $x_3$ | 0 | 0 | 2 | 1 | 0 | 2 | = | 4 |
| $x_4$ | 0 | 0 | -1 | 0 | 1 | -2 | = | 1 |
| $x_1$ | 0 | 1 | 6 | 0 | 0 | 3 | = | 9 |

| Basic Var | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| -z | 1 | 0 | -8 | -3 | 0 | 0 | = | -23 |
| $x_5$ | 0 | 0 | 1 | 0.5 | 0 | 1 | = | 2 |
| $x_4$ | 0 | 0 | 1 | 1 | 1 | 0 | = | 5 |
| $x_1$ | 0 | 1 | 3 | -1.5 | 0 | 0 | = | 3 |

# How do we find the first bfs?

- **Fact 1: If start with a basic feasible solution, we can use the simplex algorithm to find an optimal basic feasible solution.**
- **Fact 2: If we start with an LP with "<=" constraints and non-negative RHS, it is easy to find an initial bfs.**
- **How can we use these facts to find the first bfs for problem P?**

$$\begin{aligned} \max \quad & z = -3x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 4 \\ & -2x_1 + x_2 - x_3 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{aligned}$$

**Example of Problem P**

# How do we find the first bfs?

**We will create a new problem P* such that**

1. **It is easy to find a bfs for P***
2. **An optimal solution for P* is feasible for P.**

**Choose a solution *x***

**Problem P***

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + x_3 &\le 4 \\ -2x_1 + x_2 - x_3 &\le 1 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 &\ge 0 \end{aligned}$$

**and so that $x_1 + x_2 + x_3$ is as close to 4 as possible and $-2 x_1 + x_2 - x_3$ is as close to 1 as possible.**

# The Phase 1 Problem

**minimize** $y_1 + y_2$ **maximize** $v = -y_1 - y_2$

s.t.

$$x_1 + x_2 + x_3 + y_1 = 4$$

$$-2x_1 + x_2 - x_3 + y_2 = 1$$

$$x_1 \geq 0,\; x_2 \geq 0,\; x_3 \geq 0,\; y_1 \geq 0,\; y_2 \geq 0$$

**Problem P***

| -v | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | -1 | -1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 4 |
| 0 | -2 | 1 | -1 | 0 | 1 | 1 |

# Rules for creating Problem P*

**Assume we start with equality constraints and RHS >= 0.**

**Change the equality constraints to "≤ constraints".**

**Add "artificial variables" y as slack variables.**

**Minimize $y_1 + y_2 + \ldots$**

$$\text{minimize} \quad y_1 + y_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + x_3 + y_1 \quad &= 4 \\ -2x_1 + x_2 - x_3 + \quad y_2 &= 1 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ y_1 \geq 0,\ y_2 \geq 0$$

**Problem P***

# The Phase 1 Problem in canonical form

| -v | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | -1 | -1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 4 |
| 0 | -2 | 1 | -1 | 0 | 1 | 1 |

**Add constraints 1 and 2 to the objective in order to get into canonical form.**

| -v | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | RHS |
|---|---|---|---|---|---|---|
| 1 | -1 | 2 | 0 | 0 | 0 | 5 |
| 0 | 1 | 1 | 1 | 1 | 0 | 4 |
| 0 | -2 | 1 | -1 | 0 | 1 | 1 |

# Time for a mental break

Even smart people get it wrong occasionally.

"Even considering the improvements possible... the gas turbine could hardly be considered a feasible application to airplanes because of the difficulties of complying with the stringent weight requirements."

-- US National Academy Of Science, 1940

"People have been talking about a 3,000 mile high-angle rocket shot from one continent to another, carrying an atomic bomb and so directed as to be a precise weapon... I think we can leave that out of our thinking."

-- Dr. Vannevar Bush, 1945

**Fooling around with alternating current is a waste of time. Nobody will use it, ever.**
**-- Thomas Edison**

**There is not the slightest indication that nuclear energy will be obtainable.**
**-- Albert Einstein 1932**

**Rail travel at high speed is not possible because passengers, unable to breathe, would die of asphyxia.**
**-- Dr. Dionysus Lardner, 1793-1859**

**Inventions have long since reached their limit, and I see no hope for future improvements.**
**-- Julius Frontenus, 10 AD**

# The Phase 1 Problem

| -v | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|
| 1 | -1 | 2 | 0 | 0 | 0 | 5 |
| 0 | 1 | 1 | 1 | 1 | 0 | 4 |
| 0 | -2 | 1 | -1 | 0 | 1 | 1 |

**The variables $y_1$, $y_2$, $y_3$ are called *artificial variables*.**

**Theorem.** ***There is a feasible solution for P if and only if the optimal objective value for P* is 0.***

# The next pivot

| -v | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| 1 | -1 | 2 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | -2 | 1 | -1 |

| $y_1$ | $y_2$ |
|---|---|
| 0 | 0 |
| 1 | 0 |
| 0 | 1 |

| |
|---|
| 5 |
| 4 |
| 1 |

| -v | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| 1 | 3 | 0 | 2 |
| 0 | 3 | 0 | 2 |
| 0 | -2 | 1 | -1 |

| $y_1$ | $y_2$ |
|---|---|
| 0 | -2 |
| 1 | -1 |
| 0 | 1 |

| |
|---|
| 3 |
| 3 |
| 1 |

# One more pivot till the optimum for Phase 1

| -v | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 2 | 0 | -2 | 3 |
| 0 | 3 | 0 | 2 | 1 | -1 | 3 |
| 0 | -2 | 1 | -1 | 0 | 1 | 1 |

| -v | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | -1 | -1 | 0 |
| 0 | 3/2 | 0 | 1 | 1/2 | -1/2 | 3/2 |
| 0 | -1/2 | 1 | 0 | 1/2 | 1/2 | 5/2 |

**Let P be the original linear program. Let P* be the LP after adding artificial variables. Suppose $y_j > 0$ in the optimal solution for P*, where $y_j$ is artificial. Then**

1. The problem P has no feasible solution.
2. The problem P is unbounded from above.
3. If we ignore $y_j$, the solution is feasible for P.
4. Either (1) or (2) is true.

# Phase 1, Phase 2

- **If there is a feasible solution for P, then Phase 1 ends with a feasible basis.**
- **To start Phase 2, put back the original objective function. Then put the tableau in canonical form. (The basis is almost in canonical form. But the z-row is not yet right.)**
- **Then pivot until optimal (or until there is proof of unboundedness.)**

# End of Phase 1.

| $-v$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 0 | 3/2 | 0 | 1 |
| 0 | -1/2 | 1 | 0 |

| $y_1$ | $y_2$ |
|---|---|
| -1 | -1 |
| 1/2 | -1/2 |
| 1/2 | 1/2 |

| |
|---|
| 0 |
| 3/2 |
| 5/2 |

| $-z$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| 1 | -3 | 1 | 1 |

# If the RHS is greater than 0, then the next bfs has greater objective value.

| -z | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4.5 | 6 | 0 | 0 | 0 | | 0 |
| 0 | 6 | 5 | 8 | 1 | 0 | 0 | | 60 |
| 0 | 10 | 20 | 10 | 0 | 1 | 0 | | 150 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | | 8 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.75 | 0 | -0.75 | 0 | 0 | | -45 |
| | 0.75 | 0.625 | 1 | 0.125 | | | | 7.5 |
| | 2.5 | 13.75 | | -1.25 | 1 | | | 75 |
| | 1 | | | | | 1 | | 8 |

# Is the Simplex Method Finite?

**Theorem.** ***If the objective value improves at every iteration, then every basic feasible solution is different, and the simplex method is finite.***

**Proof. Each canonical tableau is uniquely determined by choosing n basic variables out of n variables. The number of bases is at most:**

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

# If the RHS is 0, it is possible that the solution stays the same after a pivot.

| -z | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4.5 | 6 | 0 | 0 | 0 | | 0 |
| 0 | 6 | 5 | 8 | 1 | 0 | 0 | | 0 |
| 0 | 10 | 20 | 10 | 0 | 1 | 0 | | 150 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | | 8 |

| | | | | | | | | RHS |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.75 | 0 | -0.75 | 0 | 0 | | 0 |
| | 0.75 | 0.625 | 1 | 0.125 | | | | 0 |
| | 2.5 | 13.75 | | -1.25 | 1 | | | 150 |
| | 1 | | | | | 1 | | 8 |

**If one of the basic variables is 0 (RHS is 0), we say that the tableau is *degenerate*.**

# If the RHS is 0, it is possible that the objective increases.

| -z | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4.5 | 6 | 0 | 0 | 0 | | 0 |
| 0 | 6 | 5 | 8 | 1 | 0 | 0 | | 60 |
| 0 | 10 | 20 | 10 | 0 | 1 | 0 | | 150 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.75 | 0 | -0.75 | 0 | 0 | | -45 |
| | 0.75 | 0.625 | 1 | 0.125 | | | | 7.5 |
| | 2.5 | 13.75 | | -1.25 | 1 | | | 75 |
| | 1 | | | | | 1 | | 0 |

**If many bases are degenerate, it is possible for the simplex algorithm to *cycle*, that is, repeat a sequence of basic feasible solutions.**

| -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | | RHS |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.75 | -20 | 0.5 | -6 | 0 | 0 | 0 | | -3 |
| 0 | 0.25 | -8 | -1 | 9 | 1 | 0 | 0 | | 0 |
| 0 | 0.5 | -12 | -0.5 | 3 | 0 | 1 | 0 | | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | | 1 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 4 | 3.5 | -33 | -3 | 0 | 0 | | -3 |
| | 1 | -32 | -4 | 36 | 4 | | | | 0 |
| | | 4 | 1.5 | -15 | -2 | 1 | | | 0 |
| | | | 1 | | | | 1 | | 1 |

**The Klee and Minty example, which can cycle.**

Microsoft®
Excel

# Bland's Rule

- There are several ways of guaranteeing that no set of basic variables repeats.
- The simplest way of avoiding "cycling" is Bland's rule.

**Bland's Rule:**

1. Among variables that have a positive coefficient in the z-row, choose the one with least index.
2. Among rows that satisfy the min ratio rule, choose the one with least index.

**Theorem.** The simplex method with Bland's rule is finite.

# Non-degeneracy and finiteness.

**Lemma.** ***If the RHS of a tableau is positive, then the next pivot will lead to an improved objective function value.***

**If a coefficient of the RHS of a tableau is 0, the tableau is *degenerate* (and the bfs is *degenerate*). If a bfs is degenerate, it is possible that the next pivot will lead to a different basis, but the same solution.**

**Theorem.** ***If no basis is degenerate, then the simplex method is finite.***

# Alternative Optima

| | $-z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| $A_0$ | 1 | 0 | 0 | 0 | 0 | -1 | = | -2 |
| $A_1$ | 0 | 0 | 2 | 1 | 0 | -1 | = | 4 |
| $A_2$ | 0 | 0 | -1 | 0 | 1 | 2 | = | 1 |
| $A_3$ | 0 | 1 | 6 | 0 | 0 | 3 | = | 3 |

Let $x_2 = \Delta$;

$x_1 = 3 - 6\Delta$
$x_2 = \Delta$
$x_3 = 4 - 2\Delta$
$x_4 = 1 + \Delta$
$x_5 = 0$
$z = 2$

This tableau satisfies the optimality conditions.

**If a tableau satisfies the optimality conditions, and if $\bar{c}_j = 0$ for a nonbasic variable, then there may be multiple alternative optima solutions.**

Non-degeneracy guarantees that we can choose $\Delta > 0$.

# Alternative Optima and Pivoting

| | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| $A_0$ | 1 | 0 | 0 | 0 | 0 | -1 | = | -2 |
| $A_1$ | 0 | 0 | 2 | 1 | 0 | -1 | = | 4 |
| $A_2$ | 0 | 0 | -1 | 0 | 1 | 2 | = | 1 |
| $A_3$ | 0 | 1 | 6 | 0 | 0 | 3 | = | 3 |

**If a tableau satisfies the optimality conditions, and if $\bar{c}_j$ =0 for a nonbasic variable, we can pivot to get an alternative optimal bfs. (or prove that there is a ray along which the objective stays the same).**

| | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | RHS |
|---|---|---|---|---|---|---|---|---|
| $B_0 = A_0$ | 1 | 0 | 0 | 0 | 0 | -1 | = | -2 |
| $B_1 = A_1 – 2 B_3$ | 0 | -1/3 | 0 | 1 | 0 | -2 | = | 3 |
| $B_2 = A_2 + B_3$ | 0 | 1/6 | 0 | 0 | 1 | 2.5 | = | 1.5 |
| $B_3 = A_3/6$ | 0 | 1/6 | 1 | 0 | 0 | .5 | = | .5 |

# Overview

- The simplex method has been a huge success in optimization.
  - It solves linear programs efficiently
  - We can solve problems with millions of variables
  - It can be a starting point for problems that are not linear
- The simplex method requires some simple techniques to get started
  - Transformation into standard form
  - Phase 1 of the simplex algorithm
  - In practice, it requires lots of implementation care
- Degeneracy and  techniques to avoid “cycling”.
- Alternative optima