Optimization Methods in Management Science

MIT 15.053

RECITATION 8
TAs: GIACOMO NANNICINI, EBRAHIM NASRABADI

At the end of this recitation, students should be able to:

1. Derive Gomory cut from fractional rows of the Simplex tableau.

2. Have an intuitition of the geometry of Gomory cuts.

3. Understand the concept of valid cover and of minimal cover for a knapsack problem.

Problem 1

Consider the linear programming feasible region and the integer program feasible region given in the diagram to the right.
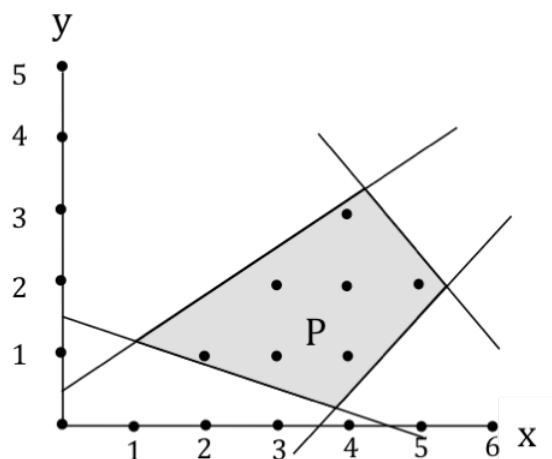

Figure 1: Sketch of the feasible region of the LP relaxation for Problem 1

Which of the following are valid inequalities for the integer program?

(a) $x \leq 4$

> **Solution.** No

(b) $x \geq 2$

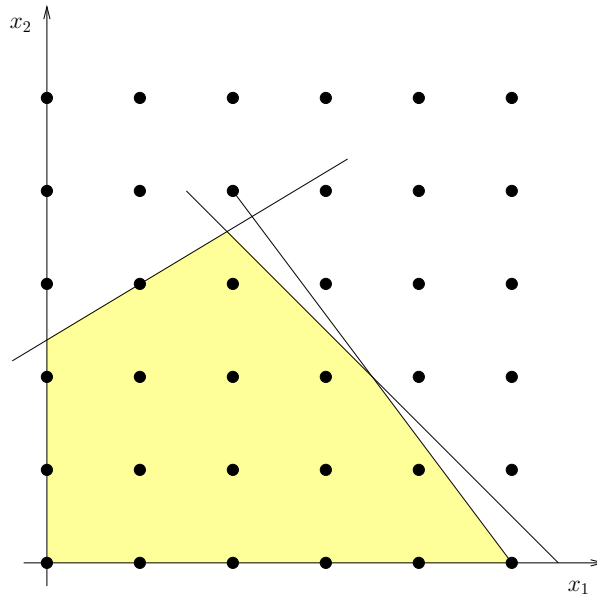> **Solution.** Yes

(c) $y \geq 1$

> **Solution.** Yes

Figure 2: Sketch of the feasible region of the LP relaxation for Part 2.A.

(d) $y \le 6$ Yes

(e) $x + y \ge 3$.

Solution. Yes

Problem 2

Consider the feasible region of an Integer Program defined by the following constraints:

$$\left.\begin{array}{rrcl}
& -3x_1 + 5x_2 & \le & 12 \\
& 4x_1 + 3x_2 & \le & 20 \\
& x_1 + x_2 & \le & 11/2 \\
\forall j = 1, 2 & x_j & \ge & 0 \\
\forall j = 1, 2 & x_j & \in & \mathbb{Z}.
\end{array}\right\} (IP)$$

(a) Sketch the feasible region of the LP relaxation of the problem. Then, determine the convex hull of the feasible integer points of (IP), first graphically, and then algebraically, using your sketch to determine the inequalities that define the convex hull. (Note: the first constraint passes through the points $(-4, 0)$ and $(1, 3)$, and the second constraint passes through the points $(5, 0)$ and $(2, 4)$. This should allow you to draw the feasible region very quickly.)

Solution. The feasible region of the LP relaxation is sketched in Figure 2. The feasible integer points are those inside the yellow region.

The convex hull of the feasible integer points is the smallest convex set that contains all of them. We can determine the set graphically as shown in Figure 3. It is sometimes also called simply the convex hull of the Integer Program.

It is not hard too see graphically that the integer hull is defined by three constraints: $-x_1 + x_2 \le 2$, $x_2 \le 3$ and $x_1 + x_2 \le 5$, together with nonnegativity $x_1 \ge 0, x_2 \ge 0$.
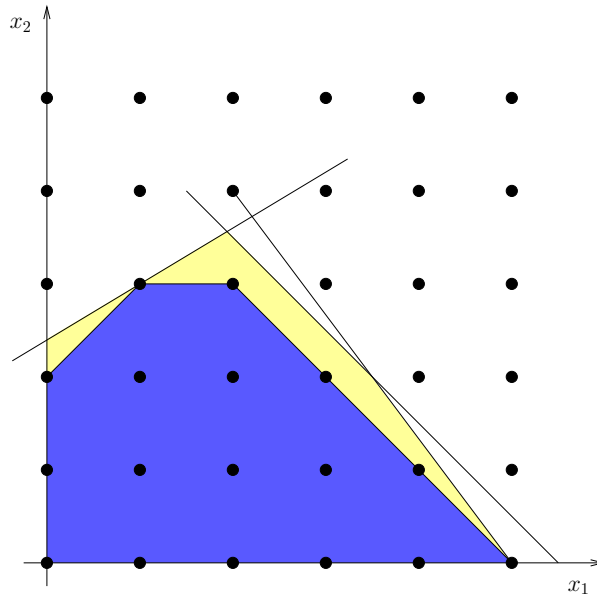
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Figure 3: Sketch of the integer hull of the integer points for Part 2.A.

(b) Let F be the feasible region of (IP). (Note: F is a set of *integer* points, it is *not* the feasible region of the LP relaxation of (IP).) Consider the set X defined as: $X = \{x_1 + x_2 \leq 11/2 : x_1, x_2 \in \mathbb{Z}\}$. What is the relationship between F and X, in terms of inclusion?

Now determine a valid inequality for the set X. Is this inequality also valid for (IP)? Justify your answer based on the relationship between F and X. (Recall the definition of valid inequality for a set: it is an inequality that does not cut off any integer feasible point of the set.)

> **Solution.**   The set X is defined by only some of the constraints that define F: it has integrality on the two variables, and the constraint $x_1 + x_2 \leq 11/2$. It is therefore clear that every point that satisfies (IP) also satisfies the constraints of X. Hence, $F \subseteq X$.
>
> Any inequality that does not cut off integer points that satisfy $x_1 + x_2 \leq 11/2$ is a valid inequality for X. We can easily determine such an inequality by noticing that the left-hand side of the inequality integer, therefore we can round down the right-hand side. We obtain $x_1 + x_2 \leq 5$. This inequality is valid for X and is valid for (IP) as well: in fact, it is one of the inequalities that define the convex hull of (IP). It is easy to show validity: because $x_1 + x_2 \leq 5$ is valid for X, it does not cut off any point in X. Because X is a relaxation of F (that is, $F \subseteq X$), $x_1 + x_2 \leq 5$ cannot cut off any point in F. It follows that the inequality is valid for (IP).

Problem 3

We want to solve an integer program (which is a maximization problem) using the Gomory cutting plane technique. We first consider the linear programming relaxation with the following initial LP tableau.

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-z)$	-2	4	1				0
s_1	2	1	1	1			8
s_2	-4	4	2		1		8
s_3	1	2	3			1	9

After three pivots, we obtain the following optimal tableau for the LP relaxation:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-z)$	0	0	$-2\frac{1}{3}$		$-\frac{2}{3}$	$-\frac{2}{3}$	$-11\frac{1}{3}$
s_1			$-1\frac{1}{2}$	1	$\frac{1}{4}$	-1	1
x_2		1	$1\frac{1}{6}$		$\frac{1}{12}$	$\frac{1}{3}$	$3\frac{2}{3}$
x_1	1		$1\frac{2}{3}$		$-\frac{1}{6}$	$\frac{1}{3}$	$1\frac{2}{3}$

(a) Derive two Gomory cuts based on the final two constraints of the second tableau.

Solution. Gomory cut from the second row: By rounding down the coefficients and the RHS, we obtain

$$x_1 + x_3 + 0s_2 + 0s_3 \leq 3,$$

By subtracting this inequality from the the second constraint in the tableau, we get the following Gomory cut:

$$x_3/3 + s_2/12 + s_3/3 \geq 2/3. \tag{1}$$

Similarly, we get the following Gomory cut from the third row:

$$2x_3/3 + 5s_2/6 + s_3/3 \geq 2/3 \tag{2}$$

(b) Does the basic feasible solution from final tableau satisfy either of the two constraints from Part (a)?

Solution. No, the current LP solution from Tableau 2 does not satisfy either of the two constraints from part (a).

Problem 4

Consider the knapsack set: $K = \{x \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$. We want to derive valid inequalities for this set.

(a) For each of the following inequalities, identify whether or not they are valid knapsack covers, and explain why.

(a) $x_4 + x_5 + x_6 \leq 2$

(b) $x_1 + x_2 + x_6 \leq 2$

(c) $x_2 + x_3 + x_6 + x_7 \leq 3$

(d) $x_2 + x_4 + x_5 + x_6 \leq 3$

(e) $x_1 + x_3 + x_4 + x_5 \leq 3$

(f) $x_2 + x_3 + x_4 + x_5 + x_6 \leq 4$

Solution. Suppose we have a knapsack constraint $\sum_{j=1}^{n} a_j x_j \leq b$ with n variables. Let $C \subseteq \{1, \ldots, n\}$. An inequality of the form $\sum_{j \in C} x_j \leq |C| - 1$ is valid if $\sum_{j \in C} a_j > b$.

We can use this definition to check validity of the inequalities above. For example, the first inequality is *not* valid: we have $C = \{4, 5, 6\}$ and $\sum_{j \in C} a_j = 5 + 5 + 4 \leq 19$. Intuitively, this can be verified as follows: the weight of items 4, 5 and 6 sums up to 14. Therefore, we cannot say that we must select at most 2 out of these 3 items: we could potentially select all of them without violating the knapsack constraint. At the same time, the second inequality is valid: we have $C = \{1, 2, 6\}$, which yields $11 + 6 + 4 > 19$. So, we can safely state that we can select at most 2 items out of the set $\{1, 2, 6\}$: otherwise, we would violate the knapsack constraint.

With the same logic, we can verify validity of all the inequalities.

(a) Not valid.

(b) Valid.

(c) Not valid.

(d) Valid.

(e) Valid.

(f) Valid.

(b) For all valid knapsack covers in Part (a), identify whether or not they are minimal.

Solution. A knapsack cover is minimal if removing any element from the cover yields a set that is not a cover. This is better explained by an example. Consider inequality 2) above: $x_1 + x_2 + x_6 \leq 2$. The cover is formed by the elements $\{1, 2, 6\}$, whose weight sums up to $21 > 19$. If we remove any element from the cover, the sum of the weights will be less than 19, hence we would not obtain another cover. It follows that the cover $\{1, 2, 6\}$ and the corresponding inequality are minimal.

Similarly, we can determine that 4) is minimal, 5) is not minimal, and 6) is not minimal.

(c) (Advanced material: understanding this question is *not* required for the Midterm.)

Consider the knapsack cover $C = \{2, 3, 4, 5\}$ that yields the inequality $x_2 + x_3 + x_4 + x_5 \leq 3$. Do you see a way of enlarging the cover while keeping the same right-hand side of the inequality? That is: can you find a valid inequality of the form $x_2 + x_3 + x_4 + x_5 + [\ldots] \leq 3$, where the $[\ldots]$ stands for one or more variables? If you can find such an inequality (called an *extended* cover), do you think that it is stronger or weaker than the initial cover? Justify your answer.

Solution. If we look at the weight of the items in the cover (i.e. the coefficient with which the items appear in the knapsack constraint), we see that x_1 has larger weight than every item in the cover. Therefore, if we can select at most 3 items out of the set C, we can also select at most 3 items out of the set $C \cup \{1\}$.

This can be proven as follows. By hypothesis, $x_2 + x_3 + x_4 + x_5 \leq 3$ is a valid cover inequality. We want to prove that $x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$ is valid. Suppose not. Then there exists a set a feasible solution to the knapsack set K such that four items out of $\{1, 2, 3, 4, 5\}$ are selected. Note that the four items with least weight out of $\{1, 2, 3, 4, 5\}$ are $2, 3, 4, 5$. Therefore, every set of four items out of $\{1, 2, 3, 4, 5\}$ has weight at least equal to the weight of $\{2, 3, 4, 5\}$. But $\{2, 3, 4, 5\}$ is a valid cover by hypothesis, which

implies that the sum of the weight of every four items out of $\{1, 2, 3, 4, 5\}$ is larger than 19. The contradiction follows.

The inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$ is an extended cover. The extended cover is at least as strong as the initial cover. In fact, because $x_1 \geq 0$, we have $x_2 + x_3 + x_4 + x_5 \leq x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$. It follows that every point that satisfies the extended cover also satisfies the initial cover: the extended cover cuts off *at least* the same points that are cut off by the initial cover. However, the converse is not necessarily true. For example, the point $x_1 = 3/4, x_2 = 3/4, x_3 = 3/4, x_4 = 3/4, x_5 = 3/4$, which is not feasible for K, is not cut off by the initial cover, but is cut off by the extended cover.

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